HIGH ENERGY SIGN-CHANGING SOLUTIONS FOR CORON'S PROBLEM

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Abstract: We study the existence of sign changing solutions to the following problem

$$\begin{cases} \Delta u + |u|^{p-1}u = 0 & \text{in} \quad \Omega_\varepsilon; \\ u = 0 & \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \tag{0.1}$$

where $p = \frac{n+2}{n-2}$ is the critical Sobolev exponent and Ω_ε is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 3$, of the form $\Omega_\varepsilon = \Omega \backslash B(0, \varepsilon)$. Here Ω is a smooth bounded domain containing the origin 0 and $B(0, \varepsilon)$ denotes the ball centered at the origin with radius $\varepsilon > 0$. We construct a new type of sign-changing solutions with high energy to problem (0.1), when the parameter ε is small enough.

1. Introduction

Let D be a smooth bounded domain in $\mathbb{R}^n$, with $n \geq 3$, and let $p = \frac{n+2}{n-2}$ be the critical Sobolev exponent, namely the exponent for which the Sobolev embedding $H_0^1(D) \hookrightarrow L^{p+1}(D)$ ceases to be compact. A problem which has been widely studied in the last 30 years concerns the existence, multiplicity, and qualitative properties for positive or sign-changing solutions to

$$\begin{cases} \Delta u + |u|^{p-1}u = 0 & \text{in} \quad D; \\ u = 0 & \text{on} \quad \partial D. \end{cases} \tag{1.1}$$

Solvability for problem (1.1) is not a trivial issue, since it strongly depends on the geometry of D. Let us briefly summarize some classical results. A direct consequence of Pohozaev's indentity [24] is that problem (1.1) has no positive solutions when the domain D is strictly star-shaped. On the other hand, if D is an annulus, then Kazdan and Warner [15] showed that solvability for problem (1.1) is restored: a solution is found as critical point of the energy functional associated to the problem, and the required compactness for the functional is obtained thanks to radial symmetry. A surprising result by Coron showed later on that symmetry is not really needed to have solvability: in his classical work [6] he proves the existence of a positive solution to (1.1) in the case in which D has a small (not necessarily symmetric) hole. In literature, the name *Coron's problem* is referred to problem (1.1), when the domain D has a hole. The result by Coron was then generalized by Bahri and Coron in [2], where the authors showed that, under the assumption that some homology group of D with coefficients in Z_2 is not trivial then problem (1.1) has at least one positive solution. Multiplicity result for positive solutions to (1.1) is obtained in [25], where the situation of a domain D with several holes is treated. On the other hand, existence and multiplicity of positive solutions has been established also in contractible domains, we refer to the works of Dancer [7], Ding [10], and Passaseo [22]- [23], among others.

The study of solutions for elliptic problems with critical nonlinearity which change sign has received the interest of several authors in the last years, see for instance [3, 4], [13, 14], and

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references therein. Here we focus our interest in existence and qualitative properties of sign-changing solutions to (1.1) for domains D which have a hole, that is in the *Coron's setting.* The first result available in literature is the one contained in [18], where a large number of sign changing solutions to (1.1) in the presence of a single small hole has been proved. To be more precise, the authors assume that the domain D has the form $D = \Omega\backslash B(0,\varepsilon)$, where Ω is a smooth bounded domain containing the origin, and it is symmetric with respect to the origin, while the hole is given by $B(0,\varepsilon)$, a round ball with radius ε. In this situation, they prove that, for any given integer k there exist $\varepsilon_k > 0$ so that, for any $\varepsilon \in (0,\varepsilon_k)$ a sign-changing solution to (1.1) exists, and it has the shape of a superposition of exactly k bubbles, centered at 0. A bubble is the function

$$U(y) = \alpha_n \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad \alpha_n = [n(n-2)]^{\frac{n-2}{4}}, \tag{1.2}$$

and it solves

$$\Delta u + u^p = 0, \quad \text{in} \quad \mathbb{R}^n. \tag{1.3}$$

It is well known that all positive solutions to (1.3) are given by the function (1.2) and any translation, and proper dilation of it, that is by

$$U(x-\xi), \quad \text{for} \quad \xi \in \mathbb{R}^n, \quad \text{and} \quad \lambda^{-\frac{n-2}{2}}U(\frac{x}{\lambda}), \quad \text{for} \quad \lambda > 0, \tag{1.4}$$

see [1, 21, 26]. The solutions found in [18] are bubble-tower, with k bubbles of alternating sign, which at main order look like

$$u_\varepsilon(x) \sim \sum_{j=1}^{k} (-1)^{j+1}\lambda_j^{-\frac{n-2}{2}} U\left(\frac{x}{\lambda_j}\right), \quad \lambda_j \sim \varepsilon^{\frac{2j-1}{2k}}, \tag{1.5}$$

as $\varepsilon \to 0^+$. The energy of these solutions has the asymptotic expansion

$$e(u_\varepsilon) = kS_n + O\left(\varepsilon^{\frac{n-2}{2}}\right), \quad \text{as} \quad \varepsilon \to 0^+, \quad \text{where} \quad S_n = \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\int_{\mathbb{R}^n} |\nabla U|^2\,dx\right). \tag{1.6}$$

We recall the explicit definition of the energy e, given by

$$e(u) = \frac{1}{2}\int_D |\nabla u|^2 - \frac{1}{p+1}\int_D |u|^{p+1}. \tag{1.7}$$

Substantial improvement of this result was obtained in [12] where the assumption of symmetry was removed. See also [17]. A different construction of sign-changing solutions for (1.1) on a domain with a small hole has been obtained in [5]: in this case, solutions look like the sum of a positive bubble centered inside the shrinking hole, and a number of sign-changing bubbles centered inside the domain, far from the hole. A common feature of the constructions in [5, 12, 17, 18] above described is that the building blocks that constitute the core of the shape of the solutions are given by the positive solutions to (1.3), which are completely classified by the functions given by (1.2) and (1.4).

The aim of this work is to produce a new, different construction of sign-changing solutions to problem (1.1), which are build upon a different limit profile. Indeed, we use as building blocks for our construction an explicit *sign-changing* solution to

$$\Delta u + |u|^{p-1}u = 0, \quad \text{in} \quad \mathbb{R}^n. \tag{1.8}$$

In [8] it is proven the existence of a sequence of finite-energy, sign-changing solutions Q_k to (1.8), with a *crown-like* shape and energy of size $(k+1)S_n(1+o(1))$, as $k \to \infty$. For any k large, Q_k

is

$$Q_k(x) = U_*(x) + \tilde{\phi}(x), \quad U_*(x) = U(x) - \sum_{j=1}^{k} U_j(x), \tag{1.9}$$

where $\tilde{\phi}$ is smaller than U_*, in some sense that we precise later. The function U is given by (1.2) and U_j are defined as

$$U_j(x) = \mu_k^{-\frac{n-2}{2}} U(\mu_k^{-1}(x - \xi_j)).$$

For any integer k large, the k points ξ_j, $j = 1, \ldots, k$ are vertices of a regular polygon of k edges, contained in the (x_1, x_2)-plane, given by

$$\xi_j = \sqrt{1 - \mu_k^2}\,(\cos\theta_j, \sin\theta_j, 0) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}, \quad \theta_j = \frac{2\pi(j-1)}{k}$$

and the parameter μ_k is defined as

$$\Big[\sum_{j=1}^{k} \frac{1}{(1 - \cos\theta_j)^{\frac{n-2}{2}}}\Big]\, \mu_k^{\frac{n-2}{2}} = 1 + O(\frac{1}{k}), \quad \text{for} \quad k \to \infty. \tag{1.10}$$

Observe that the previous definition gives

$$\mu_k \sim \begin{cases} k^{-2}, & \text{if} \;\; n \geq 4, \\ k^{-2}|\log k|^{-2}, & \text{if} \;\; n = 3. \end{cases} \tag{1.11}$$

In other words, Q_k is a *crown-like* function, with a central positive bubble, centered at the origin in $\mathbb{R}^n$, and a large number of negative copies of bubbles, centered at the vertices of a regular k-polygon sit in the $(x_1, x_2, 0, \ldots, 0)$-plane, each one with a very sharp profile, as $k \to \infty$. A property of these solutions that is central for our construction to work is that they are non-degenerate, as proved in [19]. In the class of bounded functions, the linearized operator around Q_k

$$L(\phi) = \Delta\phi + p|Q_k|^{p-1}\phi$$

has a kernel of finite dimension, equals to $3n$. Section 2 is devoted to give a precise description of the main properties of these solutions, including their non-degeneracy.

In this paper, we construct sign changing solutions to problem (1.1) using the function Q_k as main building block, thus generating a new type of sign-changing solutions, different from the ones already known in literature [5, 12, 17, 18].

To state our result, we let Ω be a smooth bounded domain in $\mathbb{R}^n$, $\varepsilon > 0$ and $\Omega_\varepsilon = \Omega \backslash B(0, \varepsilon)$. We consider the following problem

$$\begin{cases} \Delta u + |u|^{p-1}u = 0 & \text{in } \Omega_\varepsilon; \\ u = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{1.12}$$

We have the validity of the following

Theorem 1.1. *There exists a sequence of integers $(k_m)_{m\geq 1}$, with $\lim_{m\to\infty} k_m = \infty$, and a sequence $(\varepsilon_m)_{m\geq 1}$, such that for any $\varepsilon \in (0, \varepsilon_m)$, Problem (1.12) has a sign changing solution u_ε, satisfying*

$$u_\varepsilon(x) = d_\varepsilon^{-\frac{n-2}{2}} \varepsilon^{-\frac{n-2}{4}} Q_{k_m}\left(\frac{x}{\sqrt{\varepsilon} d_\varepsilon}\right)(1 + o(1))$$

where $o(1) \to 0$ uniformly as $\varepsilon \to 0$, and $d_\varepsilon \to d_0$ with d_0 is a positive constant. Moreover

$$J_\varepsilon(u_\varepsilon) = (k_m + 1)S_n + O(\varepsilon^{\frac{n-2}{2}}), \quad \textit{as} \quad \varepsilon \to 0^+,$$

where

$$J_\varepsilon(u_\varepsilon)=\frac{1}{2}\int_{\Omega_\varepsilon}|\nabla u|^2dx-\frac{1}{p+1}\int_{\Omega_\varepsilon}|u|^{p+1}dx.$$

Observe once more that the solutions predicted by Theorem 1.1 are new, and differ from the bubble towers found in [12, 18] and described in (1.5). Indeed, let k be an integer (large but fixed) for which the solution predicted by Theorem 1.1 exist. Let us denote by u_ε^1 such solution. For the same integer k, a bubble tower exists, with the shape (1.5). Let us denote by u_ε^2 this solution. One has that

$$\|u_\varepsilon^1\|_\infty \sim \frac{k^{n-2}}{\varepsilon^{\frac{n-2}{4}}},\quad \|u_\varepsilon^2\|_\infty \sim \frac{1}{\varepsilon^{\frac{n-2}{2}(1-\frac{1}{2k})}},\quad \text{while}\quad \text{as}\quad \varepsilon\to 0.$$

This paper is organized as follows. In Section 2 we recall the main properties of the functions Q_k, that are used in the rest of the paper. In Section 3, we define a first approximation to problem (1.12). We give the expansion of the energy functional at the first approximation in Section 4. The proof of Theorem 1.1 is contained in Section 5. Section 6 is devoted to solve a linear problem, and Section 7 is devoted to solve a nonlinear problem. Further properties on the functions Q_k, which are new, are reported in the Appendix A.

2. Building blocks: Sign-Changing solutions Q_k in $\mathbb{R}^n$

The solutions predicted by Theorem 1.1 are constructed as small perturbation of an initial approximation. This initial approximation is build using the entire, finite energy, sign changing solution Q_k for the problem (1.8), which are mentioned in the Introduction. The existence of such solutions is proven in [8, 9]: this is a sequence of solutions, defined for any integer k sufficiently large. If we define the energy by

$$E(u)=\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2\,dy-\frac{1}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}\,dy, \tag{2.1}$$

we have

$$E(Q_k)=\begin{cases}(k+1)\,S_n\,\left(1+O(k^{2-n})\right) & \text{if } n\geq 4\,,\\ (k+1)\,S_3\,\left(1+O(k^{-1}|\log k|^{-1}\right) & \text{if } n=3,\end{cases}$$

as $k\to\infty$, where S_n is defined in (1.6). The solution $Q=Q_k$ decays at infinity like the fundamental solution, namely

$$\lim_{|y|\to\infty}|y|^{n-2}\,Q_k(y)=\beta_n\,(1+c_k) \tag{2.2}$$

where β_n is a positive number and $c_k=O(\frac{1}{k^{n-3}})$ if $n\geq 4$, while $c_k=O(|\log k|^{-1})$, if $n=3$, as $k\to\infty$. Furthermore, the solution $Q=Q_k$ has a positive global non degenerate maximum at $y=0$. To be more precise we have

$$Q(y)=[n(n-2)]^{\frac{n-2}{4}}\left(1-\frac{n-2}{2}|y|^2+O(|y|^3)\right)\quad \text{as}\quad |y|\to 0. \tag{2.3}$$

Another property for the solution $Q=Q_k$ is that it is invariant under rotation of angle $\frac{2\pi}{k}$ in the y_1,y_2 plane, namely

$$Q(e^{\frac{2\pi}{k}}\bar y,y')=Q(\bar y,y'),\quad \bar y=(y_1,y_2),\quad y'=(y_3,\dots,y_n). \tag{2.4}$$

It is even in the y_j-coordinates, for any $j=2,\dots,n$

$$Q(y_1,\dots,y_j,\dots,y_n)=Q(y_1,\dots,-y_j,\dots,y_n),\quad j=2,\dots,n. \tag{2.5}$$

It respects invariance under Kelvin's transform:

$$Q(y) = |y|^{2-n} Q(\frac{y}{|y|^2}). \tag{2.6}$$

The function $\tilde\phi$ in (1.9) can be further decomposed. Let us introduce some cut-off functions ζ_j to be defined as follows. Let $\zeta(s)$ be a smooth function such that $\zeta(s) = 1$ for $s < 1$ and $\zeta(s) = 0$ for $s > 2$. We also let $\zeta^-(s) = \zeta(2s)$. Then we set

$$\zeta_j(y) = \begin{cases} \zeta(\, k\eta^{-1}|y|^{-2}|(y - \xi_j|y|)\,|\,) & \text{if } |y| > 1\,, \\ \\ \zeta(\, k\eta^{-1}\, |y - \xi_j|\,) & \text{if } |y| \le 1\,, \end{cases}$$

in such a way that that $\zeta_j(y) = \zeta_j(y/|y|^2)$. The function $\tilde\phi$ has the form

$$\tilde\phi \;=\; \sum_{j=1}^{k} \tilde\phi_j + \psi. \tag{2.7}$$

In the decomposition (2.7) the functions $\tilde\phi_j$, for $j \ge 1$, are defined in terms of $\tilde\phi_1$

$$\tilde\phi_j(\bar y, y') = \tilde\phi_1(e^{\frac{2\pi(j-1)}{k} i}\bar y, y'), \quad j = 1, \dots, k. \tag{2.8}$$

We have that

$$\|\psi\|_{n-2} \le Ck^{1-\frac{n}{q}} \quad \text{if} \quad n \ge 4, \quad \|\psi\|_{n-2} \le \frac{C}{\log k} \quad \text{if} \quad n = 3, \tag{2.9}$$

where $q > \frac{n}{2}$, and

$$\|\phi\|_{n-2} := \|\, (1 + |y|^{n-2})\phi\,\|_{L^\infty(\mathbb{R}^n)}\;. \tag{2.10}$$

On the other hand, if we rescale and translate the function $\tilde\phi_1$

$$\phi_1(y) = \mu^{\frac{n-2}{2}}\, \tilde\phi_1(\xi_1 + \mu y) \tag{2.11}$$

we have the validity of the following estimate for ϕ_1

$$\|\phi_1\|_{n-2} \le Ck^{-\frac{n}{q}} \quad \text{if} \quad n \ge 4, \quad \|\phi_1\|_{n-2} \le \frac{C}{k \log k} \quad \text{if} \quad n = 3. \tag{2.12}$$

The description of the solution Q in (1.9) is thus quite accurate. For later purpose, we observe that the region where Q changes sign is well understood: there exists $0 < R_1 < 1 < R_2$, positive, so that

$$Q(y) > 0, \quad \text{for} \quad |y| \le R_1, \quad |y| \ge R_2. \tag{2.13}$$

In [19], is was proved that for a sequence of integers $k_n = k$, these solutions are *non degenerate*. That is, fix one solution $Q = Q_k$ of problem (1.8) and define the linearized equation around Q as follows

$$L(\phi) = \Delta\phi + p|Q|^{p-1}\phi. \tag{2.14}$$

The invariances (2.4), (2.5), (2.6), together with the natural invariance of any solution to (1.8) under translation (if u solves (1.8) then also $u(y + \zeta)$ solves (1.8) for any $\zeta \in \mathbb{R}^n$) and under dilation (if u solves (1.8) then $\lambda^{-\frac{n-2}{2}} u(\lambda^{-1} y)$ solves (1.8) for any $\lambda > 0$) produce some *natural* functions φ in the kernel of L, namely $L(\varphi) = 0$. These are the $3n$ linearly independent functions we introduce next:

$$z_0(y) = \frac{n-2}{2} Q(y) + \nabla Q(y) \cdot y, \tag{2.15}$$

$$z_\alpha(y) = \frac{\partial}{\partial y_\alpha} Q(y), \quad \text{for} \quad \alpha = 1, \dots, n, \tag{2.16}$$

and

$$z_{n+1}(y) = -y_2 \frac{\partial}{\partial y_1} Q(y) + y_1 \frac{\partial}{\partial y_2} Q(y), \tag{2.17}$$

$$z_{n+2}(y) = -2y_1 z_0(y) + |y|^2 z_1(y), \quad z_{n+3}(y) = -2y_2 z_0(y) + |y|^2 z_2(y) \tag{2.18}$$

and, for $l = 3, \dots, n$

$$z_{n+l+1}(y) = -y_l z_1(y) + y_1 z_l(y), \quad z_{2n+l-1}(y) = -y_l z_2(y) + y_2 z_l(y). \tag{2.19}$$

One has

$$L(z_\alpha) = 0, \quad \text{for all} \quad \alpha = 0, 1, \dots, 3n-1. \tag{2.20}$$

To show (2.20), let us introduce the following operator: for any set of parameters $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$ and for any function $f : \mathbb{R}^n \to \mathbb{R}$, we define the following function

$$\Theta_A[f](x) = \lambda^{-\frac{n-2}{2}} \left| \frac{x-\xi}{|x-\xi|} - a\frac{|x-\xi|}{\lambda} \right|^{2-n} f\left(\frac{R_\theta\left(\frac{x-\xi}{\lambda} - a|\frac{x-\xi}{\lambda}|^2\right)}{|\frac{x-\xi}{|x-\xi|} - a\frac{|x-\xi|}{\lambda}|^2} \right). \tag{2.21}$$

Here and in what follows in the paper, with abuse of notation but with no ambiguity, we use the notation a to denotes a vector in $\mathbb{R}^2$, $a = (a_1, a_2) \in \mathbb{R}^2$, as well as the vector in $\mathbb{R}^n$, whose first two components are a_1, a_2, and all the other components are zero, namely $a = (a_1, a_2, 0, \cdots, 0) \in \mathbb{R}^n$.

When $f = Q = Q_k$ is the non-degenerate solution to problem (1.8), for simplicity we define

$$\Theta_A(x) = \Theta_A[Q](x). \tag{2.22}$$

In [11] it is proven that for any choice of A, the function Θ_A is still a solution of (1.8), namely

$$\Delta\Theta_A + |\Theta_A|^{p-1}\Theta_A = 0, \quad \text{in} \quad \mathbb{R}^n.$$

Observe now that

$$\left(\frac{\partial}{\partial \lambda}\Theta_{(\lambda,0,0,0)}(x)\right)_{\lambda=1} = -z_0(x), \quad \left(\frac{\partial}{\partial \xi_j}\Theta_{(1,\xi,0,0)}(x)\right)_{\xi=0} = -z_j(x), \quad j = 1, \dots, n \tag{2.23}$$

$$\left(\frac{\partial}{\partial a_1}\Theta_{(1,0,a,0)}(x)\right)_{a=0} = z_{n+2}(x), \quad \left(\frac{\partial}{\partial a_2}\Theta_{(1,0,a,0)}(x)\right)_{a=0} = z_{n+3}(x). \tag{2.24}$$

Identities (2.23) and (2.24) say that z_0 is related to the invariance of Problem (1.8) with respect to dilation $\lambda^{-\frac{n-2}{2}} Q(\lambda^{-1}y)$, z_i, $i = 1, \dots, n$, are related to the invariance of Problem (1.8) with respect to translation $Q(y + \zeta)$, z_{n+2} and z_{n+3} defined in (2.18) are related to the invariance of Problem (1.8) under Kelvin transformation (2.6). We shall see next that the function z_{n+1} defined in (2.17) is related to the invariance of Q under rotation in the (y_1, y_2) plane, while the functions defined in (2.19) are related to the invariance under rotation in the (y_1, y_l) plane and in the (y_2, y_l) plane respectively.

Let us be more precise. Denote by $O(n)$ the orthogonal group of $n \times n$ matrices P with real coefficients, so that $P^T P = I$, and by $SO(n) \subset O(n)$ the special orthogonal group of all matrices in $O(n)$ with $det P = 1$. $SO(n)$ is the group of all rotations in $\mathbb{R}^n$, it is a compact group, which can be identified with a compact set in $\mathbb{R}^{\frac{n(n-1)}{2}}$. Consider the sub group $\hat{S}$ of $SO(n)$ generated by rotations in the (x_1, x_2)-plane, in the (x_j, x_α)-plane, for any $j = 1, 2$ and $\alpha = 3, \dots, n$. We have that $\hat{S}$ is compact and can be identified with a compact manifold of dimension $2n-3$, with no boundary. In other words, there exists a smooth injective map $\chi : \hat{S} \to \mathbb{R}^{\frac{n(n-1)}{2}}$ so that $\chi(\hat{S})$

is a compact manifold of dimension $2n-3$ with no boundary and $\chi^{-1}:\chi(\hat{S})\to\hat{S}$ is a smooth parametrization of $\hat{S}$ in a neighborhood of the Identity. Thus we write

$$\theta\in\mathcal{O}:=\chi(\hat{S}),\quad R_\theta=\chi^{-1}(\theta) \tag{2.25}$$

where $\mathcal{O}$ is a compact manifold of dimension $2n-3$ with no boundary and R_θ denotes a rotation in $\hat{S}$. Let $\theta=(\theta_{12},\theta_{13},\dots,\theta_{1n},\theta_{23},\dots,\theta_{2n})$, and we write

$$R_\theta=R_{12}(\theta_{12})R_{13}(\theta_{13})R_{14}(\theta_{14})\cdots R_{1n}(\theta_{1n})R_{23}(\theta_{23})R_{24}(\theta_{24})\cdots R_{2n}(\theta_{2n}),$$

where $R_{ij}(\theta_{ij})$ is the Rotation in the $(i,j)-$plane,

$$R_{ij}(\theta_{ij})=\begin{pmatrix}1\cdots & 0 & 0\cdots 0 & 0 & \cdots 0\\ \vdots\ \ddots & \vdots & \vdots\ \ddots\ \vdots & \vdots & \ddots\ \vdots\\ 0\cdots & \cos\theta_{ij} & 0\cdots 0 & -\sin\theta_{ij} & \cdots 0\\ 0\cdots & 0 & 1\cdots 0 & 0 & \cdots 0\\ \vdots\ \ddots & \vdots & \vdots\ \ddots\ \vdots & \vdots & \ddots\ \vdots\\ 0\cdots & 0 & 0\cdots 1 & 0 & \cdots 0\\ 0\cdots & \sin\theta_{ij} & 0\cdots 0 & \cos\theta_{ij} & \cdots 0\\ \vdots\ \ddots & \vdots & \vdots\ \ddots\ \vdots & \vdots & \ddots\ \vdots\\ 0\cdots & 0 & 0\cdots 0 & 0 & \cdots 1\end{pmatrix},\qquad i<j.$$

We set

$$R_\theta=(c_{ij})_{n\times n}.$$

By a direct calculation, we have

$$\begin{aligned}c_{11}&=\cos\theta_{12}\cos\theta_{13}\cos\theta_{14}\cdots\cos\theta_{1n},\\ c_{i1}&=\sin\theta_{1i}\cos\theta_{1,i+1}\cos\theta_{1,i+2}\cdots\cos\theta_{1n},\quad i=2,3,\dots,n,\end{aligned}$$

and

$$\begin{aligned}c_{12}=&-\sin\theta_{12}\cos\theta_{23}\cos\theta_{2,4}\cdots\cos\theta_{2n}\\ &-\cos\theta_{12}\sin\theta_{13}\sin\theta_{23}\cos\theta_{24}\cdots\cos\theta_{2n}\\ &-\cos\theta_{12}\cos\theta_{13}\sin\theta_{14}\sin\theta_{24}\cos\theta_{25}\cdots\cos\theta_{2n}\\ &-\cdots\\ &-\cos\theta_{12}\cos\theta_{13}\cos\theta_{14}\cdots\cos\theta_{1,n-1}\cos\theta_{1,n-1}\sin\theta_{2,n-1}\cos\theta_{2n}\\ &-\cos\theta_{12}\cos\theta_{1,3}\cos\theta_{14}\cdots\cos\theta_{1,n-1}\cos\theta_{1,n-1}\sin\theta_{1n}\sin\theta_{2n},\end{aligned}$$

and for $i=2,3,\dots,n$,

$$\begin{aligned}c_{i2}=&\cos\theta_{1i}\sin\theta_{2i}\cos\theta_{2,i+1}\cos\theta_{2,i+2}\cdots\cos\theta_{2n}\\ &-\sin\theta_{1i}\sin\theta_{1,i+1}\sin\theta_{2,i+1}\cos\theta_{2,i+2}\cos\theta_{2,i+2}\cdots\cos\theta_{2n}\\ &-\sin\theta_{1i}\cos\theta_{1,i+1}\sin\theta_{1,i+2}\sin\theta_{2,i+2}\cos\theta_{2,i+3}\cos\theta_{2,i+2}\cdots\cos\theta_{2n}\\ &-\cdots\\ &-\sin\theta_{1i}\cos\theta_{1,i+1}\cos\theta_{1,i+2}\cdots\cos\theta_{1,n-2}\sin\theta_{1,n-1}\sin\theta_{2,n-1}\cos\theta_{2n}\\ &-\sin\theta_{1i}\cos\theta_{1,i+1}\cos\theta_{1,i+2}\cdots\cos\theta_{1,n-2}\cos\theta_{1,n-1}\sin\theta_{1,n}\sin\theta_{2n}.\end{aligned}$$

We have

$$\left(\frac{\partial}{\partial\theta_{12}}\Theta_{(1,0,\theta,0)}(x)\right)_{\theta=0}=z_{n+1}(x) \tag{2.26}$$

and, for any $l = 3, \dots, n$,

$$\left(\frac{\partial}{\partial \theta_{1l}} \Theta_{(1,0,\theta,0)}(x) \right)_{\theta=0} = z_{n+l+1}(x), \quad \left(\frac{\partial}{\partial \theta_{2l}} \Theta_{(1,0,\theta,0)}(x) \right)_{\theta=0} = z_{n+l-1}(x) \tag{2.27}$$

Thus we have the validity of (2.20) as direct consequence of (2.23), (2.24), (2.26), (2.27).

In [19] it is proven that there exists a sequence of solutions $Q = Q_k$ of the form (1.9) which are non degenerate in the sense that

$$\mathrm{Kernel}(L) = \mathrm{Span}\{z_\alpha \,:\, \alpha = 0, 1, 2, \dots, 3n-1\}, \tag{2.28}$$

or equivalently, any bounded (or any solution in $\mathcal{D}^{1,2}$) of $L(\varphi) = 0$ is a linear combination of the functions z_α, $\alpha = 0, \dots, 3n-1$. The non-degeneracy of Q is a crucial property for our construction.

3. The first approximation solution

Let $\eta > 0$ be a fixed and small number and let us introduce a set of parameters $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2n-3}$ with the properties that

$$\lambda = d\sqrt{\varepsilon}, \quad \text{with } \ \eta < d < \frac{1}{\eta}, \quad \text{for some fixed } \eta > 0, \tag{3.1}$$

$$\xi = \lambda \tau \in \mathbb{R}^n, \quad \text{with } \ |\tau| < \eta \tag{3.2}$$

$$a = \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{B} := \left\{ a = (a_1, a_2, 0, \dots, 0)^T \in \mathbb{R}^n \ : \ |a| < \frac{1}{2} \right\}, \tag{3.3}$$

and

$$\theta = (\theta_{12}, \theta_{13}, \dots, \theta_{1n}, \theta_{23}, \dots, \theta_{2n}) \in \mathcal{O} \tag{3.4}$$

where $\mathcal{O}$ is a compact manifold of dimension $2n-3$ with no boundary which was introduced in the previous section. The elements in $\mathcal{O}$ represents the invariants of any solution to (1.8) generated by rotations in the (x_1, x_2)-plane, in the (x_j, x_α)-plane, for any $j = 1, 2$ and $\alpha = 3, \dots, n$.

For any set of parameters $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$, we now introduce the function,

$$Q_A(x) = \Theta_A(R_\theta^{-1}(x - \xi)),$$

where Θ_A is defined in (2.22). More explicitly,

$$Q_A(x) = \lambda^{-\frac{n-2}{2}} \left| \frac{x-\xi}{|x-\xi|} - R_\theta a \frac{|x-\xi|}{\lambda} \right|^{2-n} Q\left(\frac{\frac{x-\xi}{\lambda} - R_\theta a |\frac{x-\xi}{\lambda}|^2}{1 - 2R_\theta a \cdot \frac{x-\xi}{\lambda} + |a|^2 |\frac{x-\xi}{\lambda}|^2} \right). \tag{3.5}$$

Observe that Q_A solves the equation in (1.12), but it is far from satisfying the boundary conditions. For this reason, we correct Q_A, introducing its projection onto $H_0^1(\Omega_\varepsilon)$. Let us define $P_\varepsilon Q_A$ to be

$$\begin{cases} -\Delta P_\varepsilon Q_A = |Q_A|^{p-1} Q_A & \text{in} \quad \Omega_\varepsilon; \\ P_\varepsilon Q_A = 0 & \text{on} \quad \partial\Omega_\varepsilon. \end{cases} \tag{3.6}$$

We next give the description of the asymptotic behavior of the projection $P_\varepsilon Q_A$ for $x \in \Omega_\varepsilon$, as $\varepsilon \to 0$. To do so, we need to introduce Green's function $G(x,y)$ of the domain, namely G satisfies

$$\begin{cases} -\Delta_x G(x,y) = \delta(x-y) & x \in \Omega; \\ G(x,y) = 0 & x \in \partial\Omega, \end{cases} \tag{3.7}$$

where $\delta(x)$ denotes the Dirac mass at the origin, and its regular part $H(x,y) := \Gamma(x-y) - G(x,y)$, where Γ denotes the fundamental solution of the Laplacian,

$$\Gamma(x) = \gamma_n |x|^{2-n}. \tag{3.8}$$

It is direct to see that

$$\begin{cases} -\Delta_x H(x,y) = 0 & x \in \Omega; \\ H(x,y) = \Gamma(x-y) & x \in \partial\Omega. \end{cases} \tag{3.9}$$

Furthermore, we introduce the function

$$F(\tau, a, \theta) := \begin{cases} Q\left(-\frac{\tau}{|\tau|^2} - R_\theta a\right) |\tau|^{2-n} & \text{if } \tau \neq 0\,, \\ \\ \lim_{|z|\to 0} Q(z) & \text{if } \tau = 0. \end{cases} \tag{3.10}$$

This is a smooth function in the set of parameters τ, a and θ satisfying (3.2), (3.3), (3.4). We have the validity of the following

Lemma 3.1. *Let $\eta > 0$ be fixed and assume that $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$ satisfies (3.1)-(3.4), with the additional assumption that $\xi = \lambda\tau$, and $\tau \neq 0$. Let*

$$R(x) := P_\varepsilon Q_A(x) - Q_A(x) + \gamma_n^{-1} \lambda^{\frac{n-2}{2}} Q(-R_\theta a) H(x,\xi) + \frac{1}{\lambda^{\frac{n-2}{2}}} F(\tau, a, \theta) \frac{\varepsilon^{n-2}}{|x|^{n-2}},$$

where F is defined in (3.10). *Then there exists a positive constant c such that for any $x \in \Omega \setminus B(0,\varepsilon)$*

$$|R(x)| \leq c\lambda^{\frac{n-2}{2}} \left[\frac{\varepsilon^{n-2}(1 + \varepsilon\lambda^{-n+1})}{|x|^{n-2}} + \lambda^2 + \frac{\varepsilon^{n-2}}{\lambda^{n-2}} \right], \tag{3.11}$$

$$|\partial_\lambda R(x)| \leq c\lambda^{\frac{n-4}{2}} \left[\frac{\varepsilon^{n-2}(1 + \varepsilon\lambda^{-n+1})}{|x|^{n-2}} + \lambda^2 + \frac{\varepsilon^{n-2}}{\lambda^{n-2}} \right], \tag{3.12}$$

$$|\partial_{\tau_i} R(x)| \leq c\lambda^{\frac{n}{2}} \left[\frac{\varepsilon^{n-2}(1 + \varepsilon\lambda^{-n})}{|x|^{n-2}} + \lambda^2 + \frac{\varepsilon^{n-2}}{\lambda^{n-1}} \right], \tag{3.13}$$

$$|\partial_{a_i} R(x)| \leq c\lambda^{\frac{n-2}{2}} \left[\frac{\varepsilon^{n-2}(1 + \varepsilon\lambda^{-n+1})}{|x|^{n-2}} + \lambda^2 + \frac{\varepsilon^{n-2}}{\lambda^{n-2}} \right], \tag{3.14}$$

$$\left|\partial_{\theta_{ij}} R(x)\right| \leq c\lambda^{\frac{n-2}{2}} \left[\frac{\varepsilon^{n-2}(1 + \varepsilon\lambda^{-n+1})}{|x|^{n-2}} + \lambda^2 + \frac{\varepsilon^{n-2}}{\lambda^{n-2}} \right]. \tag{3.15}$$

Proof. Let us introduce the scaled function $\hat{R}(y) = \lambda^{-\frac{n-2}{2}} R(\varepsilon y)$, defined for $y \in \hat{\Omega}_\varepsilon = \left(\frac{\Omega}{\varepsilon}\right) \setminus B(0,1)$. Thus $-\Delta \hat{R} = 0$ in $\hat{\Omega}_\varepsilon$. Furthermore, $\hat{\Omega}_\varepsilon \to \mathbb{R}^n \setminus B(0,1)$ as $\varepsilon \to 0$. Observe that, if $z = \frac{\varepsilon y - \xi}{\lambda}$, then

$$Q_A(\varepsilon y) = \lambda^{-\frac{n-2}{2}} |z|^{2-n} Q\left(\frac{z}{|z|^2} - R_\theta a\right).$$

For any $y \in \partial B(0,1)$ we have that

$$\hat{R}(y) = -\lambda^{-\frac{n-2}{2}} Q_A(\varepsilon y) + \gamma_n^{-1} Q(-R_\theta a) H(\varepsilon y, \xi) + \lambda^{2-n} F(\tau, a, \theta).$$

If $\xi = \lambda\tau$, and $\tau \neq 0$, then $z = -\tau + \frac{\varepsilon}{\lambda}y$, and a direct Taylor expansion gives that

$$Q_A(\varepsilon y) = \lambda^{-\frac{n-2}{2}} Q\left(-\frac{\tau}{|\tau|^2} - R_\theta a\right) |\tau|^{2-n} \left(1 + O(\frac{\varepsilon}{\lambda})\right)$$

uniformly for points $y \in \partial B(0,1)$. If $\tau = 0$, then

$$\begin{aligned} Q_A(\varepsilon y) =& \lambda^{-\frac{n-2}{2}} \left(\lim_{|z|\to 0} |z|^{2-n} Q\Big(\frac{z}{|z|^2}\Big)\right) \left(1 + O(\frac{\varepsilon}{\lambda})\right) \\ =& \lambda^{-\frac{n-2}{2}} \left(\lim_{|z|\to 0} Q(z)\right) \left(1 + O(\frac{\varepsilon}{\lambda})\right) \end{aligned}$$

Thus we get the estimates

$$|\hat{R}(y)| \leq C(1 + \frac{1}{\lambda^{n-2}} \frac{\varepsilon}{\lambda}) \quad \text{uniformly for} \quad y \in \partial B(0,1). \tag{3.16}$$

Let us now take $y \in \partial\left(\frac{\Omega}{\varepsilon}\right)$, and we have

$$\hat{R}(y) = -\lambda^{-\frac{n-2}{2}} Q_A(\varepsilon y) + \frac{\gamma_n}{|\varepsilon y - \lambda\tau|^{n-2}} Q(R_\theta a) + \frac{1}{\lambda^{n-2}} F(\tau, a, \theta) \frac{1}{|y|^{n-2}}.$$

Since $|z| = |-\tau + \frac{\varepsilon y}{\lambda}| \to \infty$, as $\varepsilon \to 0$, we get

$$|\hat{R}(y)| \leq C(\lambda^2 + (\frac{\varepsilon}{\lambda})^{n-2}) \quad \text{uniformly for } y \in \partial\left(\frac{\Omega}{\varepsilon}\right).$$

A comparison argument for harmonic functions implies that

$$|\hat{R}(y)| \leq C\left[\frac{1 + \varepsilon\lambda^{1-n}}{|y|^{n-2}} + \lambda^2 + (\frac{\varepsilon}{\lambda})^{n-2}\right].$$

This fact gives (3.11).

Let us now denote by $R_\lambda(x) = \partial_\lambda R(x)$ and define $\hat{R}_\mu(y) = \lambda^{-\frac{n-4}{2}} R(\varepsilon y)$. A direct computation shows that

$$|\hat{R}_\lambda(y)| \leq C(1 + \frac{1}{\lambda^{n-2}} \frac{\varepsilon}{\lambda}) \quad \text{uniformly for} \quad y \in \partial B(0,1),$$

and

$$|\hat{R}_\lambda(y)| \leq C(\lambda^2 + (\frac{\varepsilon}{\lambda})^{n-2}) \quad \text{uniformly for} \quad y \in \partial\left(\frac{\Omega}{\varepsilon}\right).$$

This fact gives (3.12).

Finally, let $R_i(x) = \partial_{\tau_i} R(x)$ and $\hat{R}_i(y) = \lambda^{-\frac{n}{2}} R_i(\varepsilon y)$. We get the following estimates

$$|\hat{R}_i(y)| \leq C(1 + \frac{\varepsilon}{\lambda^n}) \quad \text{uniformly for} \quad y \in \partial B(0,1),$$

and

$$|\hat{R}_i(y)| \leq C(\lambda^2 + \frac{\varepsilon^{n-2}}{\lambda^{n-1}}) \quad \text{uniformly for} \quad y \in \partial\left(\frac{\Omega}{\varepsilon}\right).$$

This fact gives (3.13). In a similar way, one gets estimates (3.14) and (3.15). □

4. The expansion of the energy

In this Section, we give the expansion of the energy function $J_\varepsilon(P_\varepsilon Q_A)$ which defined by

$$J_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} |u|^{p+1} dx.$$

We have the following result.

Proposition 4.1. *Let $\eta > 0$ be fixed, and $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$ satisfies (3.1)-(3.4). Then*

$$J_\varepsilon(P_\varepsilon Q_A) = c_1 + \frac{1}{2}\left(\gamma_n^{-2} Q(-R_\theta a)^2 H(0,0) d^{n-2} + \frac{c_2}{d^{n-2}} F(\tau, a, \theta)\right)\varepsilon^{\frac{n-2}{2}} + \Pi(d, \tau, a, \theta)\,\varepsilon^{\frac{n-1}{2}} \tag{4.1}$$

and

$$\nabla_{(d,\tau,a,\theta)} J_\varepsilon(P_\varepsilon Q_A) = \nabla_{(d,\tau,a,\theta)}\left[\frac{1}{2}\left(\gamma_n^{-2} Q(-R_\theta a)^2 H(0,0) d^{n-2} + \frac{c_2}{d^{n-2}} F(\tau, a, \theta)\right)\right]\varepsilon^{\frac{n-2}{2}} + \nabla\Pi(d, \tau, a, \theta)\,\varepsilon^{\frac{n-1}{2}} \tag{4.2}$$

as $\varepsilon \to 0$, where Π denote a smooth function of its variables, which is uniformly bounded as $\varepsilon \to 0$ for $(\lambda, \xi, a, \theta)$ satisfying (3.1)-(3.4). Here F is the function introduced in (3.10), *and c_1 and c_2 are the constants*

$$c_1 = \frac{1}{n}\int_{\mathbb{R}^n} |Q|^{p+1}\, dz, \qquad c_2 = \int_{\mathbb{R}^n} |Q|^p\, dz.$$

Proof. We compute the energy

$$J_\varepsilon(P_\varepsilon Q_A) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla P_\varepsilon Q_A|^2 dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} |P_\varepsilon Q_A|^{p+1} dx. \tag{4.3}$$

Taking into the fact that $-\Delta P_\varepsilon Q_A = |Q_A|^{p-1} Q_A$ in Ω_ε, and $P_\varepsilon Q_A = 0$ on $\partial\Omega_\varepsilon$, we have

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla P_\varepsilon Q_A|^2 dx &= \int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A P_\varepsilon Q_A dx \\ &= \int_{\Omega_\varepsilon} |Q_A|^{p+1} dx + \int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A (P_\varepsilon Q_A - Q_A) dx.\end{aligned} \tag{4.4}$$

Moreover, by a Taylor expansion, for some $t \in (0,1)$,

$$\begin{aligned}\int_{\Omega_\varepsilon} |P_\varepsilon Q_A|^{p+1} dx &= \int_{\Omega_\varepsilon} |Q_A + (P_\varepsilon Q_A - Q_A)|^{p+1} dx \\ &= \int_{\Omega_\varepsilon} |Q_A|^{p+1} dx + (p+1)\int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A (P_\varepsilon Q_A - Q_A) dx \\ &\quad + \frac{p(p+1)}{2}\int_{\Omega_\varepsilon} (tP_\varepsilon Q_A + (1-t)Q_A)^{p-1} (P_\varepsilon Q_A - Q_A)^2 dx,\end{aligned} \tag{4.5}$$

From (4.3), (4.4) and (4.5), we get

$$\begin{aligned}J_\varepsilon(P_\varepsilon Q_A) &= \frac{1}{n}\int_{\Omega_\varepsilon} |Q_A|^{p+1} dx - \frac{1}{2}\int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A (P_\varepsilon Q_A - Q_A) dx \\ &\quad - \frac{p}{2}\int_{\Omega_\varepsilon} (tP_\varepsilon Q_A + (1-t)Q_A)^{p-1} (P_\varepsilon Q_A - Q_A)^2 dx.\end{aligned} \tag{4.6}$$

We will estimate each term in the following, and then the result in Proposition is a consequence of the following Lemmas. □

Lemma 4.2. *Let $\eta > 0$ be fixed, and $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$ satisfies (3.1)-(3.4). It holds*

$$\int_{\Omega_\varepsilon} |Q_A|^{p+1} dx = \int_{\mathbb{R}^n} |Q|^{p+1}\, dz + \varepsilon^{\frac{n}{2}} \Pi(d, \tau, a, \theta), \tag{4.7}$$

where Π is a smooth function of its variables, which is uniformly bounded as $\varepsilon \to 0$ for $(\lambda, \xi, a, \theta)$ satisfying (3.1)-(3.4).

Proof. We decompose it as

$$\int_{\Omega_\varepsilon} |Q_A|^{p+1} dx = \int_{\varepsilon<|x|<\delta} |Q_A|^{p+1} dx + \int_{|x|>\delta} |Q_A|^{p+1} dx$$

for some $\delta > 0$ fixed and small. In the region $\varepsilon < |x| < \delta$ we introduce the change of variables $y = \frac{x-\xi}{\lambda} = \frac{x}{\lambda} - \tau$, so that

$$\int_{\varepsilon<|x|<\delta} |Q_A|^{p+1}\, dx = \int_{B_{1,\varepsilon}} \left[\left| \frac{y}{|y|} - R_\theta a |y| \right|^{2-n} |Q| \left(\frac{\frac{y}{|y|^2} - R_\theta a}{|\frac{y}{|y|^2} - R_\theta a|^2} \right) \right]^{p+1} dy,$$

where $B_{1,\varepsilon} := B(-\tau, \frac{\delta}{\lambda}) \setminus B(-\tau, \frac{\varepsilon}{\lambda})$. Since $|w|^{2-n} Q(\frac{w}{|w|^2}) = Q(w)$, and using the change of variables $z = \frac{y}{|y|^2}$, we then have

$$\begin{aligned}
\int_{\varepsilon<|x|<\delta} |Q_A|^{p+1}\, dx &= \int_{B_{1,\varepsilon}} \left[|y|^{2-n} |Q| \left(\frac{y}{|y|^2} - R_\theta a \right) \right]^{p+1} dy \\
&= \int_{B_{1,\varepsilon}} |y|^{2n} \left[|Q| \left(\frac{y}{|y|^2} - R_\theta a \right) \right]^{p+1} dy \\
&= \int_{B_{2,\varepsilon}} |Q\left(z - R_\theta a\right)|^{p+1}\, dz = \int_{\mathbb{R}^n} |Q|^{p+1}\, dz + \lambda^n O(1),
\end{aligned}$$

where $B_{2,\varepsilon} := B(\frac{-\tau}{|\tau|^2}, \frac{\lambda}{\varepsilon}) \setminus B(\frac{-\tau}{|\tau|^2}, \frac{\lambda}{\delta})$ if $\tau \neq 0$, and $B_{2,\varepsilon} := B(0, \frac{\lambda}{\varepsilon}) \setminus B(0, \frac{\lambda}{\delta})$ if $\tau = 0$. Thus, the above estimate holds true for a generic function $O(1)$ of the parameters (d, τ, a, θ), which is uniformly bounded as $\varepsilon \to 0$. On the other hand, in the set $|x| > \delta$ we have that $\left| \int_{|x|>\delta} |Q_A|^{p+1}\, dx \right| \leq C\lambda^n$. This concludes the proof of the Lemma. □

Lemma 4.3. *Let $\eta > 0$ be fixed, and $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$ satisfies (3.1)-(3.4). It holds, as $\varepsilon \to 0$,*

$$\begin{aligned}
\int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A (P_\varepsilon Q_A - Q_A) dx = &-\gamma_n^{-2} Q(-R_\theta a)^2 H(0,0) \lambda^{n-2} \\
&- \left(\frac{\varepsilon}{\lambda}\right)^{n-2} F(\tau, a, \theta) \int_{\mathbb{R}^n} |Q|^p\, dz + \Pi(d, \tau, a, \theta)\, \varepsilon^{\frac{n-1}{2}},
\end{aligned} \tag{4.8}$$

where Π is a smooth function of its variables, which is uniformly bounded as $\varepsilon \to 0$ for $(\lambda, \xi, a, \theta)$ satisfying (3.1)-(3.4). Here F is the function introduced in (3.10).

Proof. By Lemma 3.1, we have

$$\int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A (P_\varepsilon Q_A - Q_A) dx = -\gamma_n^{-1} \lambda^{\frac{n-2}{2}} Q(-R_\theta a) \int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A H(x, \xi) dx$$

$$-\frac{\varepsilon^{n-2}}{\lambda^{\frac{n-2}{2}}}F(\tau,a,\theta)\int_{\Omega_\varepsilon}|Q_A|^{p-1}Q_A\frac{dx}{|x|^{n-2}}+\int_{\Omega_\varepsilon}|Q_A|^{p-1}Q_A R(x)dx$$

$$:= I_1+I_2+I_3. \tag{4.9}$$

To estimate I_1, we write

$$\int_{\Omega_\varepsilon}|Q_A|^{p-1}Q_A H(x,\xi)dx=\int_{\varepsilon<|x|<\delta}|Q_A|^{p-1}Q_A H(x,\xi)dx+\int_{\Omega\cap|x|>\delta}|Q_A|^{p-1}Q_A H(x,\xi)dx,$$

for some positive, small and fixed δ. In the first region, the function H is smooth, and in particular it has bounded derivatives. Thus, by Taylor expansions, we get

$$\begin{aligned}&\int_{\varepsilon<|x|<\delta}|Q_A|^{p-1}Q_A H(x,0)dx=H(0,0)\left(\int_{\varepsilon<|x|<\delta}|Q_A|^{p-1}Q_A dx\right)\\&+O(\int_{\varepsilon<|x|<\delta}|Q_A|^{p-1}Q_A|x|dx)+\lambda O(\int_{\varepsilon<|x|<\delta}|Q_A|^{p-1}Q_A dx).\end{aligned}$$

Taking the change of variables $y=\frac{x-\xi}{\lambda}$, using the invariance of Q under Kelvin transform, and using the change of variables $z=\frac{y}{|y|^2}$, we get

$$\begin{aligned}\int_{\varepsilon<|x|<\delta}|Q_A|^{p-1}Q_A\,dx&=\lambda^{\frac{n-2}{2}}\int_{B_{1,\varepsilon}}\left[|y|^{2-n}|Q|\left(\frac{y}{|y|^2}-R_\theta a\right)\right]^p dy\\&=\lambda^{\frac{n-2}{2}}\int_{B_{1,\varepsilon}}|y|^{-(n+2)}\left[|Q|\left(\frac{y}{|y|^2}-R_\theta a\right)\right]^p dy\\&=\lambda^{\frac{n-2}{2}}\int_{B_{2,\varepsilon}}\frac{1}{|z|^{n-2}}\left|Q\left(z-R_\theta a\right)\right|^p dz\\&=\lambda^{\frac{n-2}{2}}\int_{\mathbb{R}^n}\frac{1}{|z+R_\theta a|^{n-2}}\left|Q\right|^p dz+O(\lambda^{\frac{n+2}{2}}),\end{aligned}$$

as $\varepsilon\to0$. Here again $B_{1,\varepsilon}:=B(-\tau,\frac{\delta}{\lambda})\setminus B(-\tau,\frac{\varepsilon}{\lambda})$, while $B_{2,\varepsilon}:=B(\frac{-\tau}{|\tau|^2},\frac{\lambda}{\varepsilon})\setminus B(\frac{-\tau}{|\tau|^2},\frac{\lambda}{\delta})$, if $\tau\neq0$, and $B_{2,\varepsilon}:=B(0,\frac{\lambda}{\varepsilon})\setminus B(0,\frac{\lambda}{\delta})$ if $\tau=0$. Recall now that

$$Q(-R_\theta a)=\gamma_n\int_{\mathbb{R}^n}\frac{1}{|z+R_\theta a|^{n-2}}\left|Q\right|^p dz$$

Thus we get

$$\int_{\varepsilon<|x|<\delta}|Q_A|^{p-1}Q_A dx=\gamma_n^{-1}\lambda^{\frac{n-2}{2}}Q(-R_\theta a)+O(\lambda^{\frac{n+2}{2}}). \tag{4.10}$$

On the other hand, using again the change of variables $y=\frac{x-\xi}{\lambda}$ one finds directly that $\int_{|x|<\delta}|Q_A|^{p-1}Q_A\,|x|dx=O(\lambda^{\frac{n}{2}})$. On the other hand, we observe that, in the region where $|x|>\delta$, one has

$$|Q_A(x)|\leq C\lambda^{\frac{n-2}{2}},\quad\text{for all}\quad x\in\Omega,\quad |x|>\delta, \tag{4.11}$$

where the constant C is independent of ε. Indeed, to prove (4.11), we start with the observation that, in the region under consideration, one has

$$|Q_A(x)|\leq c|Q_{\bar A}(x)|,\quad\text{where}\quad \bar A=(\lambda,0,a,\theta),$$

for some constant c, independent of ε. Now, in the region under consideration $|Q_A(x)|\leq C\lambda^{\frac{n-2}{2}}$. Thus the validity of (4.11) follows, and we find

$$I_1=-\gamma_n^{-2}\lambda^{n-2}Q(-R_\theta a)^2H(0,0)+O(\lambda^{n-1}). \tag{4.12}$$

Let us now estimate I_2. We split the integral as follows

$$\int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A \frac{1}{|x|^{n-2}} dx = \int_{\varepsilon<|x|<\delta} |Q_A|^{p-1} Q_A \frac{1}{|x|^{n-2}} dx + \int_{\Omega\cap|x|>\delta} |Q_A|^{p-1} Q_A \frac{1}{|x|^{n-2}} dx.$$

Using again (4.11), we see that $\int_{\Omega\cap|x|>\delta} |Q_A|^{p-1} Q_A \frac{1}{|x|^{n-2}} dx = O(\lambda^{\frac{n+2}{2}})$. Using the invariance of Q under Kelvin transform, and using the changes of variables, first $y = \frac{x-\xi}{\lambda}$ and then $z = \frac{y}{|y|^2}$, we get

$$\begin{aligned}\int_{\Omega_\varepsilon\cap|x|<\delta} |Q_A|^{p-1} Q_A \frac{1}{|x|^{n-2}} dx &= \lambda^{-\frac{n-2}{2}} \int_{B_{1,\varepsilon}} \left[|y|^{2-n} |Q| \left(\frac{y}{|y|^2} - R_\theta a\right)\right]^p \frac{1}{|y|^{n-2}} dy \\ &= \lambda^{-\frac{n-2}{2}} \int_{B_{1,\varepsilon}} |y|^{-2n} \left[|Q| \left(\frac{y}{|y|^2} - R_\theta a\right)\right]^p dy \\ &= \lambda^{-\frac{n-2}{2}} \int_{B_{2,\varepsilon}} |Q\,(z - R_\theta a)|^p\, dz = \lambda^{-\frac{n-2}{2}} \left(\int_{\mathbb{R}^n} |Q|^p\, dz + O(\lambda^2)\right). \qquad (4.13)\end{aligned}$$

Then

$$I_2 = -\frac{\varepsilon^{n-2}}{\lambda^{n-2}} \left(F(\tau, a, \theta) \int_{\mathbb{R}^n} |Q|^p\, dz + O(\lambda^2)\right). \qquad (4.14)$$

We conclude with the estimate for I_3. We use the result in Lemma 3.1, and in particular estimate (3.11), to get

$$|I_3| = |\int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A R(x) dx| \le c\varepsilon^{\frac{1}{2}} (|I_1| + |I_2|), \qquad (4.15)$$

for some positive constant c. This concludes the proof of Lemma. □

Lemma 4.4. *Under the same assumptions as in Proposition 4.1, it holds*

$$\int_{\Omega_\varepsilon} (tP_\varepsilon Q_A + (1-t)Q_A)^{p-1} (P_\varepsilon Q_A - Q_A)^2 dx = \Pi(d, \tau, a, \theta)\, \varepsilon^{\frac{n-1}{2}}. \qquad (4.16)$$

Proof. Using (4.6), we have

$$\begin{aligned}&|\int_{\Omega_\varepsilon} (tP_\varepsilon Q_A + (1-t)Q_A)^{p-1} (P_\varepsilon Q_A - Q_A)^2 dx| \\ &\le c \int_{\Omega_\varepsilon} |Q_A|^{p-1} \left(\lambda^{n-2} + \frac{\varepsilon^{2(n-2)}}{\lambda^{n-2}} \frac{1}{|x|^{2(n-2)}}\right) dx. \qquad (4.17)\end{aligned}$$

Direct computations give

$$\lambda^{n-2} \int_{\Omega_\varepsilon} |Q_A|^{p-1} dx = O(\lambda^{n+2}), \qquad (4.18)$$

and

$$\frac{\varepsilon^{2(n-2)}}{\lambda^{n-2}} \int_{\Omega_\varepsilon} |Q_A|^{p-1} \frac{1}{|x|^{2(n-2)}} dx = O(\frac{\varepsilon^{2(n-2)}}{\lambda^{2(n-2)}}). \qquad (4.19)$$

Then (4.16) follows from (4.17) to (4.19). □

We conclude this section with the proof of (4.2). More precisely, we prove

$$\partial_d J_\varepsilon(P_\varepsilon Q_A) = \partial_d \left[\frac{1}{2} \left(\gamma_n^{-2} Q(-R_\theta a)^2 H(0,0) d^{n-2} + \frac{c_2}{d^{n-2}} F(\tau, a, \theta)\right)\right] \varepsilon^{\frac{n-2}{2}}$$

$$+\,\Pi(d,\tau,a,\theta)\,\varepsilon^{\frac{n-1}{2}}, \tag{4.20}$$

as $\varepsilon \to 0$, where Π is a smooth function of the variables (d,τ,a,θ), which is uniformly bounded as $\varepsilon \to 0$ for (λ,ξ,a,θ) satisfying (3.1)-(3.4). The estimates for the other derivatives can be obtain in a similar way.

Proof of (4.20):

We have

$$\begin{aligned}\partial_d J_\varepsilon(P_\varepsilon Q_A) =&\partial_d\left(\frac{1}{2}\int_{\Omega_\varepsilon}|\nabla P_\varepsilon Q_A|^2dx-\frac{1}{p+1}\int_{\Omega_\varepsilon}|P_\varepsilon Q_A|^{p+1}dx\right)\\ =&\int_{\Omega_\varepsilon}\nabla P_\varepsilon Q_A\nabla(\partial_d(P_\varepsilon Q_A))dx-\int_{\Omega_\varepsilon}|P_\varepsilon Q_A|^p\partial_d(P_\varepsilon Q_A)dx\end{aligned}$$

Since the function $P_\varepsilon Q_A$ satisfies (3.6), we find

$$\partial_d J_\varepsilon(P_\varepsilon Q_A)=-\int_{\Omega_\varepsilon}\left[|P_\varepsilon Q_A|^p-|Q_A|^{p-1}Q_A\right]\partial_d(P_\varepsilon Q_A)dx.$$

By a Taylor expansion, for some $t\in(0,1)$,

$$\begin{aligned}|P_\varepsilon Q_A|^p =&|Q_A+(P_\varepsilon Q_A-Q_A)|^p=|Q_A|^{p-1}Q_A+p|Q_A|^{p-2}Q_A(P_\varepsilon Q_A-Q_A)\\ &+\frac{p(p-1)}{2}\left(tP_\varepsilon Q_A+(1-t)Q_A\right)^{p-2}\left(P_\varepsilon Q_A-Q_A\right)^2.\end{aligned}$$

Then

$$\begin{aligned}\partial_d J_\varepsilon(P_\varepsilon Q_A)=&-\int_{\Omega_\varepsilon}\left[p|Q_A|^{p-2}Q_A(P_\varepsilon Q_A-Q_A)\right]\partial_d\big(Q_A+(P_\varepsilon Q_A-Q_A)\big)dx\\ &-\frac{p(p-1)}{2}\int_{\Omega_\varepsilon}\left(tP_\varepsilon Q_A+(1-t)Q_A\right)^{p-2}\left(P_\varepsilon Q_A-Q_A\right)^2\partial_d\big(Q_A+(P_\varepsilon Q_A-Q_A)\big)dx\\ =&-\int_{\Omega_\varepsilon}\left[p|Q_A|^{p-2}Q_A(P_\varepsilon Q_A-Q_A)\right]\partial_d Q_A dx\\ &-\int_{\Omega_\varepsilon}\left[p|Q_A|^{p-2}Q_A(P_\varepsilon Q_A-Q_A)\right]\partial_d\big(P_\varepsilon Q_A-Q_A\big)dx\\ &-\frac{p(p-1)}{2}\int_{\Omega_\varepsilon}\left(tP_\varepsilon Q_A+(1-t)Q_A\right)^{p-2}\left(P_\varepsilon Q_A-Q_A\right)^2\partial_d\big(Q_A+(P_\varepsilon Q_A-Q_A)\big)dx\\ =&-\int_{\Omega_\varepsilon}\left[p|Q_A|^{p-2}Q_A(P_\varepsilon Q_A-Q_A)\right]\partial_d Q_A dx\\ &+O\left(\int_{\Omega_\varepsilon}|Q_A|^{p-2}Q_A(P_\varepsilon Q_A-Q_A)^2dx\right)\\ =&-\int_{\Omega_\varepsilon}\partial_d\left[|Q_A|^p\right](P_\varepsilon Q_A-Q_A)dx+O\left(\int_{\Omega_\varepsilon}|Q_A|^{p-2}Q_A(P_\varepsilon Q_A-Q_A)^2dx\right)\\ =&\partial_d\left(-\int_{\Omega_\varepsilon}|Q_A|^p(P_\varepsilon Q_A-Q_A)dx\right)+\int_{\Omega_\varepsilon}|Q_A|^p\partial_d(P_\varepsilon Q_A-Q_A)dx\\ &+O\left(\int_{\Omega_\varepsilon}|Q_A|^{p-2}Q_A(P_\varepsilon Q_A-Q_A)^2dx\right).\end{aligned} \tag{4.21}$$

From Lemma 4.3, we have that

$$\partial_d\left(-\int_{\Omega_\varepsilon}|Q_A|^p(P_\varepsilon Q_A-Q_A)dx\right)$$

$$=\partial_d\left(\gamma_n^{-2}Q(-R_\theta a)^2H(0,0)d^{n-2}+\frac{c_2}{d^{n-2}}F(\tau,a,\theta)\right)\varepsilon^{\frac{n-2}{2}}+\partial_d\Pi(d,\tau,a,\theta)\,\varepsilon^{\frac{n-1}{2}}$$

$$=(n-2)d^{-1}\left(\gamma_n^{-2}Q(-R_\theta a)^2H(0,0)d^{n-2}-\frac{c_2}{d^{n-2}}F(\tau,a,\theta)\right)\varepsilon^{\frac{n-2}{2}}+\partial_d\Pi(d,\tau,a,\theta)\,\varepsilon^{\frac{n-1}{2}} \tag{4.22}$$

Moreover, for the second term in (4.21), by Lemma 3.1, we have

$$\begin{aligned}
&\int_{\Omega_\varepsilon}|Q_A|^p\partial_d(P_\varepsilon Q_A-Q_A)dx\\
=&\int_{\Omega_\varepsilon}|Q_A|^p\partial_d\left[-\gamma_n^{-1}\lambda^{\frac{n-2}{2}}Q(-R_\theta a)H(x,\xi)-\frac{\varepsilon^{n-2}}{\lambda^{\frac{n-2}{2}}}F(\tau,a,\theta)\frac{1}{|x|^{n-2}}+R(x)\right]dx\\
=&\int_{\Omega_\varepsilon}|Q_A|^p\partial_\lambda\left[-\gamma_n^{-1}\lambda^{\frac{n-2}{2}}Q(-R_\theta a)H(x,\xi)-\frac{\varepsilon^{n-2}}{\lambda^{\frac{n-2}{2}}}F(\tau,a,\theta)\frac{1}{|x|^{n-2}}+R(x)\right]\frac{\partial\lambda}{\partial d}dx\\
=&\frac{n-2}{2}\varepsilon^{\frac12}\int_{\Omega_\varepsilon}|Q_A|^p\left[-\gamma_n^{-1}\lambda^{\frac{n-2}{2}-1}Q(-R_\theta a)H(x,\xi)+\frac{\varepsilon^{n-2}}{\lambda^{\frac{n-2}{2}+1}}F(\tau,a,\theta)\frac{1}{|x|^{n-2}}\right]dx\\
&+\varepsilon^{\frac12}\int_{\Omega_\varepsilon}|Q_A|^p\partial_\lambda R(x)dx\\
&\quad\text{since}\quad\lambda=\sqrt{\varepsilon}d\\
=&\frac{n-2}{2}d^{-1}\int_{\Omega_\varepsilon}|Q_A|^p\left[-\gamma_n^{-1}\lambda^{\frac{n-2}{2}}Q(-R_\theta a)H(x,\xi)+\frac{\varepsilon^{n-2}}{\lambda^{\frac{n-2}{2}}}F(\tau,a,\theta)\frac{1}{|x|^{n-2}}\right]dx\\
&+\varepsilon^{\frac12}\int_{\Omega_\varepsilon}|Q_A|^p\partial_\lambda R(x)dx\\
=&\frac{n-2}{2}d^{-1}\left[I_1-I_2\right]+\varepsilon^{\frac12}\int_{\Omega_\varepsilon}|Q_A|^p\partial_\lambda R(x)dx.
\end{aligned}$$

where I_1 and I_2 are defined in (4.9), with

$$I_1-I_2=-\left(\gamma_n^{-2}Q(-R_\theta a)^2H(0,0)d^{n-2}-\frac{c_2}{d^{n-2}}F(\tau,a,\theta)\right)\varepsilon^{\frac{n-2}{2}}+\Pi(d,\tau,a,\theta)\,\varepsilon^{\frac{n-1}{2}},$$

and from Lemma 3.1, we have $|\partial_\lambda R(x)|\le c\lambda^{-1}|R(x)|$, then by (4.15), we get

$$\begin{aligned}
|\varepsilon^{\frac12}\int_{\Omega_\varepsilon}|Q_A|^p\partial_\lambda R(x)dx|\le&c\lambda^{-1}\varepsilon^{\frac12}\int_{\Omega_\varepsilon}|Q_A|^pR(x)dx=cd\int_{\Omega_\varepsilon}|Q_A|^pR(x)dx\\
\le&c\varepsilon^{\frac12}(|I_1|+|I_2|).
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{\Omega_\varepsilon}|Q_A|^p\partial_d(P_\varepsilon Q_A-Q_A)dx\\
=&-\frac{n-2}{2}d^{-1}\left(\gamma_n^{-2}Q(-R_\theta a)^2H(0,0)d^{n-2}-\frac{c_2}{d^{n-2}}F(\tau,a,\theta)\right)\varepsilon^{\frac{n-2}{2}}+\Pi(d,\tau,a,\theta)\,\varepsilon^{\frac{n-1}{2}}.
\end{aligned} \tag{4.23}$$

Lastly, using Lemma 3.1, as a computation in Lemma 4.3, we have

$$O\left(\int_{\Omega_\varepsilon}|Q_A|^{p-2}Q_A(P_\varepsilon Q_A-Q_A)^2dx\right)=\Pi(d,\tau,a,\theta)\,\varepsilon^{\frac{n-1}{2}}, \tag{4.24}$$

as $\varepsilon\to0$, where Π is a smooth function of the variables (d,τ,a,θ), which is uniformly bounded as $\varepsilon\to0$ for (λ,ξ,a,θ) satisfying (3.1)-(3.4).

Therefore, by (4.21), (4.22), (4.23) and (4.24), we obtain

$$\partial_dJ_\varepsilon(P_\varepsilon Q_A)=\frac{n-2}{2}d^{-1}\left(\gamma_n^{-2}Q(-R_\theta a)^2H(0,0)d^{n-2}-\frac{c_2}{d^{n-2}}F(\tau,a,\theta)\right)\varepsilon^{\frac{n-2}{2}}+\partial_d\Pi(d,\tau,a,\theta)\,\varepsilon^{\frac{n-1}{2}}$$

$$=\partial_d\left[\frac{1}{2}\left(\gamma_n^{-2}Q(-R_\theta a)^2H(0,0)d^{n-2}+\frac{c_2}{d^{n-2}}F(\tau,a,\theta)\right)\varepsilon^{\frac{n-2}{2}}\right]+\partial_d\Pi(d,\tau,a,\theta)\,\varepsilon^{\frac{n-1}{2}}.$$

That is, (4.20) holds.

5. Scheme of the proof

By the change of variable,

$$v(y)=\varepsilon^{\frac{1}{p-1}}u(\sqrt{\varepsilon}y). \tag{5.1}$$

Problem (1.12) has a solution u if and only if v solves the following problem

$$\begin{cases}\Delta v+|v|^{p-1}v=0, & \text{in } D_\varepsilon;\\ v=0, & \text{on } \partial D_\varepsilon,\end{cases} \tag{5.2}$$

where $D_\varepsilon:=\frac{\Omega_\varepsilon}{\sqrt{\varepsilon}}=\frac{\Omega}{\sqrt{\varepsilon}}\backslash B(0,\sqrt{\varepsilon})$.

In expanded variable, the solution that we are looking for looks like

$$v(y)=V(y)+\phi(y),\qquad \text{where}\ \ V(y)=\varepsilon^{\frac{1}{p-1}}P_\varepsilon Q_A(\sqrt{\varepsilon}y), \tag{5.3}$$

where $P_\varepsilon Q_A$ is defined in (3.6). We observe that the function V is nothing but the projection onto $H_0^1(D_\varepsilon)$ of the function $\varepsilon^{\frac{1}{p-1}}Q_A(\sqrt{\varepsilon}y)$. We also observe that, if $A=(\lambda,\xi,a,\theta)$, then

$$\varepsilon^{\frac{1}{p-1}}Q_A(\sqrt{\varepsilon}y)\equiv Q_{\tilde{A}}(y),\quad \text{with}\quad \tilde{A}=(d,d\tau,a,\theta)$$

since $\lambda=d\sqrt{\varepsilon}$ and $\xi=\lambda\tau$, where Q_A is given in (3.5).

Rewriting the result contained in Lemma 3.1, we see that as $\varepsilon\to 0$,

$$V(y)=Q_{\tilde{A}}(y)+\varepsilon^{\frac{n-2}{2}}(1+\frac{1}{|y|^{n-2}})\Xi_{\tilde{A}}(y), \tag{5.4}$$

uniformly on compact sets of D_ε. Here $\Xi_{\tilde{A}}(y)$ is a smooth function, which is uniformly bounded for $y\in D_\varepsilon$, as $\varepsilon\to 0$, and for sets of parameters $\tilde{A}$ satisfying (3.1)-(3.4).

In terms of ϕ, problem (5.2) becomes

$$L(\phi)=-N(\phi)-E,\quad \text{in } D_\varepsilon,\quad \phi=0,\quad \text{on } \partial D_\varepsilon, \tag{5.5}$$

where

$$L(\phi)=\Delta\phi+pV^{p-1}\phi,\quad N(\phi)=(V+\phi)^p-V^p-pV^{p-1}\phi, \tag{5.6}$$

and

$$E=V^p-|Q_{\tilde{A}}|^{p-1}Q_{\tilde{A}},\quad \text{with}\quad \tilde{A}=(d,d\tau,a,\theta). \tag{5.7}$$

Consider the following functions, for any $j=0,1,2,\ldots,3n-1$,

$$Z_j(y)=\varepsilon^{\frac{1}{p-1}}\widetilde{Z}_j(\sqrt{\varepsilon}y),\quad y\in D_\varepsilon,\quad \widetilde{Z}_j(x)=\Theta_A[z_j](x), \tag{5.8}$$

where Θ_A is the operator defined in (2.21). Observe that

$$Z_j(y)=\Theta_{\tilde{A}}[z_j](y). \tag{5.9}$$

In order to solve problem (5.5), we first consider the linear problem. Let $\eta>0$ be fixed as in (3.1), and assume that the set of parameters $A=(\lambda,\xi,a,\theta)\in\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^2\times\mathbb{R}^{2n-3}$ satisfies

(3.1)-(3.4). Given a function h, we consider the problem of finding a function ϕ and real numbers c_j, $j = 0, 1, 2, \dots, 3n-1$ such that

$$
\begin{cases} L(\phi) = h + \sum_{j=0,1,2,\dots,3n-1} c_j V^{p-1} Z_j, & \text{in} \quad D_\varepsilon; \\ \phi = 0, \qquad \text{on} \quad \partial D_\varepsilon; \\ \int_{D_\varepsilon} V^{p-1} Z_j \phi dy = 0, & \text{for all } j = 0, 1, 2, \dots, 3n-1. \end{cases} \tag{5.10}
$$

In order to perform an invertibility theory for L subject to the above orthogonality conditions, we introduce some proper weighted L^∞-norms. We start with for

$$
\|\psi\|_{**} = \sup_{y \in I_\varepsilon} ||y|^{n-2} \psi(y)| + \sup_{y \in O_\varepsilon} |(1 + |y|^4) \psi(y)|, \tag{5.11}
$$

where

$$
I_\varepsilon := \{y \in D_\varepsilon, |y| < 1\}, \quad O_\varepsilon := \{y \in D_\varepsilon, |y| > 1\}. \tag{5.12}
$$

This L^∞ weighted norm, which allows singularity at 0, is suitable to estimate the right hand side h in (5.10). The estimate of $\|E\|_{**}$, where E is the function defined in (5.7), is crucial for our argument, as it will become clear later on. We claim that there exists a positive constant C, independent of ε, so that

$$
\|E\|_{**} \le C \varepsilon^{\frac{n-2}{2}}. \tag{5.13}
$$

Let us consider first $y \in D_\varepsilon$, with $|y| > 1$. Using the result in Lemma 3.1 and a Taylor expansion, in combination with (5.8) and (2.2), we immediately see that

$$
|E(y)| \le C \varepsilon^{\frac{n-2}{2}} |Q_{\tilde{A}}(y)|^{p-1} \le C \frac{\varepsilon^{\frac{n-2}{2}}}{1 + |y|^4}.
$$

Let us now consider the region $y \in D_\varepsilon$, and $|y| < 1$. In this region, the function E can be estimated as follows

$$
|E(y)| \le C \left(\frac{\varepsilon^{\frac{n-2}{2}}}{|y|^{n-2}} \right)^p \le C \frac{\varepsilon^{\frac{n-2}{2}}}{|y|^{n-2}},
$$

since in the region we are considering one has $|y| > \sqrt{\varepsilon}$. With this, (5.13) is proven. We now introduce an appropriate norm to estimate a solutions to (5.10). This norms depends on the dimension of the space. For a function ψ defined on D_ε, we define

$$
\begin{aligned} \|\psi\|_* = & \sup_{y \in I_\varepsilon} \left[||y|^\alpha \psi(y)| + ||y|^{\alpha+1} D\psi(y)| \right] \\ & + \sup_{y \in O_\varepsilon} \left[|(1 + |y|^\beta) \psi(y)| + |(1 + |y|^{\beta+1}) D\psi(y)| \right], \end{aligned} \tag{5.14}
$$

where

$$
\alpha = \begin{cases} n-4 & \text{if} \quad n \ge 5 \\ \sigma & \text{if} \quad n = 4 \end{cases}, \quad \beta = \begin{cases} 2 & \text{if} \quad n \ge 5 \\ 2 - \sigma & \text{if} \quad n = 4 \end{cases} \tag{5.15}
$$

for some $\sigma > 0$, and

$$
\|\psi\|_* = \sup_{y \in D_\varepsilon} \left[|(1 + |y|) \psi(y)| + |(1 + |y|^2) D\psi(y)| \right], \tag{5.16}
$$

if $n = 3$.

Equation (5.10) is solved in the following proposition, whose proof is postponed to Section 6.

Proposition 5.1. *Let $\eta > 0$ be fixed as in (3.1), and assume that the set of parameters $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$ satisfies (3.1)-(3.4). Then there are numbers $\varepsilon_0 > 0$, $C > 0$, such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in C^\alpha(\bar{D}_\varepsilon)$, problem (5.10) admits a unique solution $\phi := T_\varepsilon(h)$. Moreover,*

$$\|T_\varepsilon(h)\|_* \leq C\|h\|_{**}, \quad |c_j| \leq C\|h\|_{**}, \tag{5.17}$$

and

$$\|\nabla_{(d,\tau,a,\theta)}\phi\|_* \leq C\|h\|_{**}. \tag{5.18}$$

Based on the results in Proposition 5.1, a fixed point argument allows us to solve the nonlinear problem of finding a function ϕ and constants c_j solutions to

$$\begin{cases} L(\phi) = -[N(\phi) + E] + \sum\limits_{j=0,1,\dots,3n-1} c_j V^{p-1} Z_j, & \text{in} \quad D_\varepsilon; \\ \phi = 0, & \text{on} \quad \partial D_\varepsilon; \\ \int_{D_\varepsilon} V^{p-1} Z_j \phi dy = 0, \quad \text{for all } j = 0, 1, 2, \dots, 3n-1. \end{cases} \tag{5.19}$$

The solvability of problem (5.19) is established in next Proposition, whose proof is postponed to Section 7.

Proposition 5.2. *Assume the conditions of Proposition 5.1 are satisfied. Then there are numbers $\varepsilon_0 > 0$, $C > 0$, such that for all $0 < \varepsilon < \varepsilon_0$, there exists a unique solution $\phi = \phi(d, a, \theta)$ to problem (5.19). Moreover, the map $(d, \tau, a, \theta) \to \phi(d, \tau, a, \theta)$ is of class C^1 for $\|\cdot\|_*$ norm, and*

$$\|\phi\|_* \leq C\varepsilon^{\frac{n-2}{2}}, \tag{5.20}$$

and

$$\|\nabla_{d,\tau,a,\theta}\phi\|_* \leq \varepsilon^{\frac{n-2}{2}}. \tag{5.21}$$

After problem (5.10) has been solved, we find a solution to problem (5.5), if we can find a point (d, τ, a, θ) such that coefficients c_j in (5.10) satisfy

$$c_j = 0 \quad \text{for all } j = 0, 1, 2, \dots, 3n-1. \tag{5.22}$$

For notational convenience, we introduce the set

$$\mathcal{A} := \{(d, \tau, a, \theta) : \text{conditions (3.1)–(3.4) are satisfied}\} \subset \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}. \tag{5.23}$$

We now introduce the finite dimensional restriction $F_\varepsilon(d, \tau, a, \theta) : \mathcal{A} \to \mathbb{R}$, given by

$$F_\varepsilon(d, \tau, a, \theta) = I_\varepsilon\left(V(y) + \phi(y)\right), \tag{5.24}$$

with V defined by (5.3) and ϕ is the unique solution to problem (5.19) given by Proposition 5.2, and I_ε is the energy functional associated to problem (5.2), given by

$$I_\varepsilon(v) = \frac{1}{2}\int_{D_\varepsilon} |\nabla v|^2 dy - \frac{1}{p+1}\int_{D_\varepsilon} |v|^{p+1} dy. \tag{5.25}$$

Lemma 5.3. *If (d, τ, a, θ) is a critical point of F_ε, then $v(y) = V(y) + \phi(y)$ is a solution of problem (5.2).*

Proof. We claim that if (d, τ, a, θ) if a critical point for F_ε, then We first differentiate F_ε with respect to d, then we have

$$DI_\varepsilon(V+\phi)[\frac{\partial}{\partial d}Q_{\tilde{A}} + o(1)] = 0, \quad DI_\varepsilon(V+\phi)[\frac{\partial}{\partial \tau_i}Q_{\tilde{A}} + o(1)] = 0, \quad i = 1, \dots, n, \tag{5.26}$$

$$DI_\varepsilon(V+\phi)[\frac{\partial}{\partial \theta_{12}}Q_{\tilde{A}} + o(1)] = 0, \quad DI_\varepsilon(V+\phi)[\frac{\partial}{\partial a_j}Q_{\tilde{A}} + o(1)] = 0, \quad j = 1, 2,$$

$$DI_\varepsilon(V+\phi)[\frac{\partial}{\partial\theta_{1l}}Q_{\tilde A}+o(1)]=0, DI_\varepsilon(V+\phi)[\frac{\partial}{\partial\theta_{2l}}Q_{\tilde A}+o(1)]=0, \quad l=3,\dots,n.$$

Let us assume the validity of these equalities. From (5.19), we have

$$DI_\varepsilon(V+\phi)[Z_i+o(1)]=\sum_j c_j\int_{D_\varepsilon}V^{p-1}Z_j[Z_i+o(1)]dy \tag{5.27}$$

where

$$Z_0=\frac{\partial}{\partial d}Q_{\tilde A},\quad Z_j=\frac{\partial}{\partial\tau_i}Q_{\tilde A},\quad j=1,\dots,n,\quad Z_{n+1}=\frac{\partial}{\partial\theta_{12}}Q_{\tilde A}$$

$$Z_{n+2}=\frac{\partial}{\partial a_1}Q_{\tilde A},\quad Z_{n+3}=\frac{\partial}{\partial a_2}Q_{\tilde A}$$

and, for $l=3,\dots,n$,

$$Z_{n+l+1}=\frac{\partial}{\partial\theta_{1l}}Q_{\tilde A},\quad Z_{2n+l-1}=\frac{\partial}{\partial\theta_{2l}}Q_{\tilde A}.$$

Using (2.23), (2.24), (2.26), (2.27), a direct computation gives

$$\int_{D_\varepsilon}V^{p-1}Z_jZ_idy=\begin{cases}\int_{\mathbb{R}^n}|Q|^{p-1}(y)z_i^2(y)dy+O(\varepsilon^{\frac{n}{n-2}}) & \text{if } i=j;\\ \int_{\mathbb{R}^n}|Q|^{p-1}(y)z_1(y)z_{n+2}(y)dy+O(\varepsilon^{\frac{n}{n-2}}) & \text{if } i=1,\ j=n+2;\\ \int_{\mathbb{R}^n}|Q|^{p-1}(y)z_2(y)z_{n+3}(y)dy+O(\varepsilon^{\frac{n}{n-2}}) & \text{if } i=2,\ j=n+3;\\ O(\varepsilon^{\frac{n}{n-2}}) & \text{otherwise,}\end{cases}$$

where the functions z_j are the ones defined in (2.15), (2.16), (2.17), (2.18), (2.19). Therefore, the condition $\nabla_{(d,\tau,a,\theta)}F_\varepsilon(d,\tau,a,\theta)=0$ give the $3n$ conditions

$$DI_\varepsilon(V+\phi)[Z_j]=0,\quad j=0,\dots,3n-1,$$

that give necessarily that $c_j=0$ for all $j=0,\dots,3n-1$. This concludes the proof of the Lemma. We shall now prove (5.26). Since the arguments are similar, we prove the first formula in (5.26). Observe that

$$\frac{\partial}{\partial d}F_\varepsilon(d,\tau,a,\theta)=DI_\varepsilon(V+\phi)[\frac{\partial}{\partial d}V+\frac{\partial}{\partial d}\phi].$$

From Lemma 3.1 and (5.8),

$$\frac{\partial}{\partial d}V(y)=\frac{\partial}{\partial d}Q_{\tilde A}(y)+\varepsilon^{\frac{n-2}{2}}(1+\frac{1}{|y|^{n-2}})\Theta_{\tilde A}(y),$$

where $\Theta_{\tilde A}(y)$ is uniformly bounded as $\varepsilon\to0$. Now, observe that

$$\frac{\partial}{\partial d}Q_{\tilde A}(y)=d^{-\frac{n-4}{2}}\left|\frac{y-d\tau}{|y-d\tau|}-R_\theta a\frac{|y-d\tau|}{d}\right|^{2-n}z_0\left(\frac{\frac{y-d\tau}{d}-R_\theta a|\frac{y-d\tau}{d}|^2}{1-2R_\theta a\cdot\frac{y-d\tau}{d}+|a|^2|\frac{y-d\tau}{d}|^2}\right).$$

Taking into account that $\|\frac{\partial}{\partial d}\phi\|_*=o(1)$, as $\varepsilon\to0$, we get that $\frac{\partial}{\partial d}F_\varepsilon=DI_\varepsilon(V+\phi)[\frac{\partial}{\partial d}Q_{\tilde A}+o(1)]$, as $\varepsilon\to0$. □

Lemma 5.4. *Assume the conditions of Proposition 5.1 are satisfied. Then we have the following expansion*

$$F_\varepsilon(d,\tau,a,\theta)-I_\varepsilon(V)=o(\varepsilon^{\frac{n-2}{2}})\Theta,$$

where Θ is C^1 uniformly bounded, independent of ε.

Proof. By a Taylor expansion and the fact that $DI_\varepsilon(V+\phi)[\phi]=0$, we have

$$F_\varepsilon(d,\tau,a,\theta)-I_\varepsilon(V)=I_\varepsilon(V+\phi)-I_\varepsilon(V)=\int_0^1 D^2I(V+t\phi)[\phi,\phi]t\,dt$$
$$=\int_0^1\int_{D_\varepsilon}\left[|\nabla\phi|^2-p(V+t\phi)^{p-1}\phi^2\right]t\,dt.$$

From (5.19), we have

$$\begin{aligned}F_\varepsilon(d,\tau,a,\theta)-I_\varepsilon(V)&=\int_0^1\int_{D_\varepsilon}\left(p\left[V^{p-1}-(V+t\phi)^{p-1}\right]\phi^2+[N(\phi)+E]\,\phi dy\right.\\&\le C\int_{D_\varepsilon}|V^{p-1}-(V+\phi)^{p-1}|\phi^2\,dy+\int_{D_\varepsilon}|E|\,|\phi|\,dy+\int_{D_\varepsilon}|N(\phi)|\,|\phi|\,dy\\&=o(\varepsilon^{\frac{n-2}{2}})\,\Theta,\end{aligned}\tag{5.28}$$

uniformly with respect to (d,τ,a,θ) in the considered region, where Θ is uniformly bounded, independent of ε. Here we used the facts $\|E\|_*\le C\varepsilon^{\frac{n-2}{2}}$ and $\|\phi\|_*\le C\varepsilon^{\frac{n-2}{2}}$.

By a similarly way, using the facts $\|\nabla_{(d,\tau,a,\theta)}E\|_*\le C\varepsilon^{\frac{n-2}{2}}$ and $\|\partial_{(d,\tau,a,\theta)}\phi\|_*\le C\varepsilon^{\frac{n-2}{2}}$, we can obtain

$$\nabla_{(d,\tau,a,\theta)}\left(F_\varepsilon(d,\tau,a,\theta)-I_\varepsilon(V)\right)=o(\varepsilon^{\frac{n-2}{2}})\Theta.$$

This ends the proof of Lemma. □

Proof of Theorem 1.1. By Lemma 5.3, we know that $u(\sqrt{\varepsilon}y)=\varepsilon^{-\frac{1}{p-1}}(V(y)+\phi(y))$ is a solution to problem (1.12) if and only if (d,τ,a,θ) is a critical point of $F_\varepsilon(d,\tau,a,\theta)$. So we have to prove the existence of the critical point of $F_\varepsilon(d,\tau,a,\theta)$. We observe that, under the change of variables (5.1), we have $I_\varepsilon(v)=J_\varepsilon(u)$. From Lemma 5.4, Proposition 4.1, (4.1) and (4.2) we find

$$F_\varepsilon(d,\tau,a,\theta)=c_1+\Psi(d,\tau,a,\theta)\varepsilon^{\frac{n-2}{2}}+o(\varepsilon^{\frac{n-2}{2}})\Theta(d,\tau,a,\theta),\tag{5.29}$$

where Ψ is defined as

$$\Psi(d,\tau,a,\theta)=\frac{1}{2}\left[\gamma_n^{-2}Q(-R_\theta a)^2H(0,0)d^{n-2}+\frac{c_2}{d^{n-2}}F(\tau,a,\theta)\right],$$

with F given in (3.10) and Θ is a smooth function of its variables, which is uniformly bounded, together with its first derivatives, as $\varepsilon\to 0$ for (λ,ξ,a,θ) satisfying (3.1)-(3.4). Thus our result is proven provided we find a critical point, stable under C^1 perturbation, of the function Ψ. Recall that

$$A=(\lambda,\xi,a,\theta)\in\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{2n-3},\quad \lambda=d\sqrt{\varepsilon}\text{ and }\xi=\lambda\tau,$$

with d,τ,a,θ satisfying (3.1)-(3.4).

Firstly, we observe that

$$\partial_d\Psi(d,\tau,a,\theta)=\frac{n-2}{2}\gamma_n^{-2}Q(-R_\theta a)^2H(0,0)d^{n-3}-\frac{n-2}{2}\frac{c_2}{d^{n-1}}F(\tau,a,\theta).$$

We have that $\partial_d\Psi(d_0,\tau,a,\theta)=0$ with $d_0=\left(\frac{c_2F(\tau,a,\theta)}{\gamma_n^{-2}Q(-R_\theta a)^2H(0,0)}\right)^{\frac{1}{2n-4}}$, and $\partial^2_{dd}\Psi(d_0,\tau,a,\theta)>0$. Thus Ψ has a unique non-degenerate minimum point d_0 with minimum value

$$\Psi(d_0,\tau,a,\theta)=\gamma_n^{-1}\sqrt{c_2H(0,0)}Q(-R_\theta a)\sqrt{F(\tau,a,\theta)}.\tag{5.30}$$

From (2.3), we can fix $r>0$ so that

$$\frac{1}{4}\le Q(x)\le Q(0),\qquad\text{for all }|x|<r,\tag{5.31}$$

and $Q(x)$ has a local maximum in $B(0,r)$, attained at $x=0$. By the definition of F in (3.10), for $\tau \neq 0$, we have

$$\begin{aligned} F(\tau,a,\theta) =& Q\left(-\frac{\tau}{|\tau|^2}+R_\theta a\right)|\tau|^{2-n} \\ & \text{set } y=-\frac{\tau}{|\tau|^2}+R_\theta a \\ =& Q(y)|y-R_\theta a|^{n-2} \\ =& Q(y)|y|^{n-2}-Q(y)|y|^{n-4}yR_\theta a+O(|a|^2). \end{aligned}$$

Noting that $|y|\to\infty$ when $|\tau|\to 0$, thus from (2.2),

$$F(\tau,a,\theta)\to \beta_n(1+c_k)+o(|a|) \quad \text{as} \quad \tau\to 0,$$

where $c_k\to 0$ as $k\to\infty$. Moreover, for fix a and θ,

$$M:=\max_{\tau\in B(0,\eta)} F(\tau,a,\theta)>0 \tag{5.32}$$

is attained at point in $B(0,\eta)$. We define the following sets

$$\Lambda=(\eta,\eta^{-1})\times B(0,\eta)\times B(0,\frac{1}{2})\times\mathcal{O},$$

where $\eta>0$ is fixed as in (3.1), and $\mathcal{O}$ is a compact manifold of dimension $2n-3$ with no boundary, and the close connected subset of Λ given by

$$B=[2\eta,2d_0]\times\left\{\tau\in B(0,\eta): F(\tau,a,\theta)\geq\frac{M}{2}\right\}\times\overline{B(0,1/4)}\times\mathcal{O}.$$

Let Γ be the class of all continuous functions $\phi: B\to\Lambda$. Define the min-max value

$$c:=\inf_{\phi\in\Gamma}\sup_{z\in B}\Psi(\phi(z)). \tag{5.33}$$

From (5.30), (5.31) and (5.32), we find that

$$\frac{1}{4\sqrt{2}}\gamma_n^{-1}\sqrt{c_2H(0,0)}\sqrt{M}\leq c\leq\gamma_n^{-1}\sqrt{c_2H(0,0)}\sqrt{M}Q(0).$$

By construction, c is a min-max value for Ψ. This value is topologically non trivial, and it persists small C^1 perturbation of the functional. This in particular implies that also the functional F_ε defined in (5.29) admits a critical point and this concludes the proof of our result. □

6. The linear problem: proof of Proposition 5.1

Proof of Proposition 5.1: The first part of the proof consists in establishing the priori estimate (5.17). We do it by contradiction: assume that there exists a sequence $\varepsilon=\varepsilon_l\to 0$ such that there are functions ϕ_ε and h_ε such that

$$\begin{cases} L(\phi_\varepsilon)=h_\varepsilon+\sum\limits_{j=0,1,\dots,3n-1}c_jV^{p-1}Z_j, & \text{in} \quad D_\varepsilon; \\ \phi_\varepsilon=0, & \text{on} \quad \partial D_\varepsilon; \\ \int_{D_\varepsilon}V^{p-1}Z_j\phi_\varepsilon dy=0, \quad \text{for all } j=0,1,2,\dots,3n-1, & \end{cases} \tag{6.1}$$

for certain constants c_j, depending on ε, with $\|h_\varepsilon\|_{**}\to 0$ while $\|\phi_\varepsilon\|_*$ remains bounded away from 0 as $\varepsilon\to 0$.

We first establish the slightly weaker assertion that

$$\|\phi_\varepsilon\|_\rho\to 0$$

with $\rho > 0$ a small fixed number, where

$$\|\psi\|_\rho = \sup_{y \in I_\varepsilon} \left[||y|^{\alpha+\rho} \psi(y)| + ||y|^{\alpha+1+\rho} D\psi(y)| \right] + \sup_{y \in O_\varepsilon} \left[|(1+|y|^{\beta-\rho})\psi(y)| + |(1+|y|^{\beta+1-\rho}) D\psi(y)| \right],$$

if $n \geq 4$, and

$$\|\psi\|_\rho = \sup_{y \in D_\varepsilon} \left[|(1+|y|^{1-\rho})\psi(y)| + |(1+|y|^{2-\rho}) D\psi(y)| \right],$$

if $n = 3$. To do this, we assume the opposite, so that with no loss of generality we may take $\|\phi_\varepsilon\|_\rho = 1$.

We claim that

$$c_j \to 0, \quad \text{as} \quad \varepsilon \to 0. \tag{6.2}$$

Testing equation (6.4) against Z_i, integrating by parts twice we get that

$$\sum c_j \int_{D_\varepsilon} V^{p-1} Z_j Z_i dy = \int_{D_\varepsilon} [\Delta Z_i + pV^{p-1} Z_i] \phi_\varepsilon dy - \int_{\partial D_\varepsilon} Z_i \partial_\nu \phi_\varepsilon - \int_{D_\varepsilon} h_\varepsilon Z_i dy. \tag{6.3}$$

We claim that

$$\int_{D_\varepsilon} V^{p-1} Z_j Z_i dy = \begin{cases} \int_{\mathbb{R}^n} |Q|^{p-1}(y) z_i^2(y) dy + O(\varepsilon^{\frac{n}{n-2}}) & \text{if } \ i = j; \\ \int_{\mathbb{R}^n} |Q|^{p-1}(y) z_1(y) z_{n+2}(y) dy + O(\varepsilon^{\frac{n}{n-2}}) & \text{if } \ i = 1, \ j = n+2; \\ \int_{\mathbb{R}^n} |Q|^{p-1}(y) z_2(y) z_{n+3}(y) dy + O(\varepsilon^{\frac{n}{n-2}}) & \text{if } \ i = 2, \ j = n+3; \\ O(\varepsilon^{\frac{n}{n-2}}) & \text{otherwise,} \end{cases} \tag{6.4}$$

$$\int_{D_\varepsilon} [\Delta Z_i + pV^{p-1} Z_i] \phi_\varepsilon = o(1) \|\phi_\varepsilon\|_\rho, \quad \text{and} \quad \int_{\partial D_\varepsilon} Z_i \partial_\nu \phi_\varepsilon = o(1) \|\phi_\varepsilon\|_\rho \tag{6.5}$$

$$\left| \int_{D_\varepsilon} h_\varepsilon Z_i \right| \leq C \|h_\varepsilon\|_{**}. \tag{6.6}$$

Thus, we conclude that

$$|c_j| \leq C \|h_\varepsilon\|_{**} + o(1) \|\phi_\varepsilon\|_\rho \tag{6.7}$$

from which (6.2) readily follows.

Proof of (6.4). From (5.8) and (5.9), we observe that

$$\int_{D_\varepsilon} V^{p-1} Z_j Z_i dy = \int_{D_\varepsilon} Q_{\tilde{A}}^{p-1} \Theta_{\tilde{A}}[z_j] \Theta_{\tilde{A}}[z_i] \, dy + O(\varepsilon^{\frac{n}{n-2}}),$$

where $\tilde{A} = (d, d\tau, a, \theta)$. Using first the change of variable $z = \frac{y - d\tau}{d}$, and then the change of variables $\eta = \frac{z}{|z|^2}$, we have

$$\begin{aligned} \int_{D_\varepsilon} Q_{\tilde{A}}^{p-1} \Theta_{\tilde{A}}[z_j] \Theta_{\tilde{A}}[z_i] \, dy &= \int \left| \frac{z}{|z|} - R_\theta a |z| \right|^{-2n} \left(Q^{p-1} z_j z_i \right) \left(\frac{z - R_\theta a |z|^2}{|\frac{z}{|z|} - R_\theta a |z||^2} \right) dz \\ &= \int |z|^{-2n} \left| \frac{z}{|z|^2} - R_\theta a \right|^{-2n} \left(Q^{p-1} z_j z_i \right) \left(\frac{\frac{z}{|z|^2} - R_\theta a}{|\frac{z}{|z|^2} - R_\theta a|^2} \right) dz \\ &= \int \left(Q^{p-1} z_j z_i \right) (\eta - R_\theta a) \, d\eta \end{aligned}$$

$$= \int_{\mathbb{R}^n} Q^{p-1} z_j z_i + O(\varepsilon^{\frac{n}{n-2}}).$$

The last equality follows from Lemma 8.1 in the Appendix. This concludes the proof of (6.4).

Proof of (6.5). We start from the first estimate. Let $g_i = \Delta Z_i + pV^{p-1}Z_i$. By definition of Z_i, we have $g_i = p\left(V^{p-1} - Q_{\tilde{A}}^{p-1}\right) Z_i$. A close analysis of the functions z_j in (2.15), (2.16), (2.17), (2.18), (2.19) gives that, for some constant C,

$$|z_j(y)| \le \frac{C}{1+|y|^{n-2}}, \quad y \in \mathbb{R}^n.$$

From Lemma 3.1, we see that

$$V(y) = Q_{\tilde{A}}(y) - \gamma_n^{-1} Q(-R_\theta a) H(\sqrt{\varepsilon} y, \xi)\varepsilon^{\frac{n-2}{2}} - F(\tau, a, \theta)\frac{\varepsilon^{\frac{n-2}{2}}}{|y|^{n-2}} + R(y)$$

with

$$|R(y)| \le c\varepsilon^{\frac{n-2}{2}}\left[\frac{\varepsilon^{\frac{n-2}{2}}(1+\varepsilon\lambda^{-n+1})}{|y|^{n-2}} + \lambda^2 + \varepsilon^{\frac{n-2}{2}}\right].$$

Thus we have the following estimate

$$\left|[\Delta Z_i + pV^{p-1}Z_i]\phi_\varepsilon\right| \le CQ_{\tilde{A}}^{p-2}\left(\varepsilon^{\frac{n-2}{2}} + \frac{\varepsilon^{\frac{n-2}{2}}}{|y|^{n-2}}\right)|Z_i\phi_\varepsilon|. \tag{6.8}$$

To estimate $\int_{D_\varepsilon} g_i\phi$, we estimate separately $\int_{I_\varepsilon} g_i\phi$, and $\int_{O_\varepsilon} g_i\phi$. Consider first the case $p \ge 2$. In dimensions 5 and 6, one has

$$\begin{aligned}\left|\int_{I_\varepsilon}[\Delta Z_i + pV^{p-1}Z_i]\phi_\varepsilon\right| &\le C\varepsilon^{\frac{n-2}{2}}\|\phi\|_\rho \int_{I_\varepsilon}\frac{1}{|y|^{2n-6+\rho}}\,dy\\ &\le C\varepsilon^{\frac{n-2}{2}}\varepsilon^{-\frac{n}{2}+3-\frac{\sigma}{2}-\frac{\rho}{2}}\|\phi_\varepsilon\|_\rho\int_{\sqrt{\varepsilon}<|y|<1}\frac{1}{|y|^{n-\sigma}}\,dy\\ &\le C\varepsilon^{1-\frac{\sigma}{2}-\frac{\rho}{2}}\|\phi\|_\rho.\end{aligned}$$

Moreover,

$$\left|\int_{O_\varepsilon}[\Delta Z_i + pV^{p-1}Z_i]\phi_\varepsilon\right| \le C\varepsilon^{\frac{n-2}{2}}\|\phi\|_\rho\int_{O_\varepsilon}\frac{1}{1+|y|^{6-\rho}}\,dy = \varepsilon^{2-\frac{\rho}{2}}\|\phi\|_\rho.$$

In analogous way, one has

$$\left|\int_{I_\varepsilon}[\Delta Z_i + pV^{p-1}Z_i]\phi_\varepsilon\right| \le C\begin{cases}\varepsilon^{2-\frac{\rho}{2}-\frac{\sigma}{2}}\|\phi\|_\rho & \text{if } n=4;\\ \\ \varepsilon^{\frac{1}{2}}\|\phi\|_\rho & \text{if } \ n=3,\end{cases}$$

and

$$\left|\int_{O_\varepsilon}[\Delta Z_i + pV^{p-1}Z_i]\phi_\varepsilon\right| \le C\begin{cases}\varepsilon\|\phi\|_\rho & \text{if } n=4;\\ \\ \varepsilon^{\frac{1}{2}}\|\phi\|_\rho & \text{if } \ n=3,\end{cases}$$

as $\varepsilon \to 0$, in dimensions 4 and 3. Similar estimates hold also in dimensions 4 and 3. Thus the first estimate in (6.5) holds true in dimensions 3 to 6.

Let us consider now $n \ge 7$, that is $p < 2$. Define $R_\varepsilon = \{y \in D_\varepsilon \,:\, |Q_{\tilde{A}}(y)| \le \varepsilon\}$. We have

$$\left|[\Delta Z_i + pV^{p-1}Z_i]\phi_\varepsilon\right| \le C\varepsilon^{p-1}Z_j|\phi_\varepsilon|, \quad \text{in} \quad R_\varepsilon$$

and

$$\left|[\Delta Z_i + pV^{p-1}Z_i]\phi_\varepsilon\right| \le C|Q_{\tilde A}|^{p-1}\varepsilon^{-1}\left(\varepsilon^{\frac{n-2}{2}} + \frac{\varepsilon^{\frac{n-2}{2}}}{|y|^{n-2}}\right)|Z_j\phi_\varepsilon|, \quad \text{in} \quad D_\varepsilon \setminus R_\varepsilon$$

Thus, we get

$$\left|\int_{I_\varepsilon\cap R_\varepsilon} g_i\phi\right| \le C\varepsilon^{p-1}\|\phi\|_\rho, \quad \left|\int_{O_\varepsilon\cap R_\varepsilon} g_i\phi\right| \le C\varepsilon^{p-1}\|\phi\|_\rho,$$

and

$$\left|\int_{I_\varepsilon\cap R_\varepsilon^c} g_i\phi\right| \le C\varepsilon^{1-\frac{\rho}{2}-a}\|\phi\|_\rho, \quad \left|\int_{O_\varepsilon\cap R_\varepsilon^c} g_i\phi\right| \le C\varepsilon^{\frac{n-4}{2}}\|\phi\|_\rho,$$

for some $a > 0$ small. Thus we get the validity of the first estimate in (6.5). Let us discuss now the second estimate in (6.5). We write

$$\int_{\partial D_\varepsilon} Z_i\frac{\partial\phi}{\partial\nu} = \int_{\partial D_\varepsilon\cap I_\varepsilon} Z_i\frac{\partial\phi}{\partial\nu} + \int_{\partial D_\varepsilon\cap O_\varepsilon} Z_i\frac{\partial\phi}{\partial\nu}.$$

We observe that

$$\left|\int_{\partial D_\varepsilon\cap I_\varepsilon} Z_i\frac{\partial\phi}{\partial\nu}\right| \le C\|\phi\|_\rho\int_{\partial B(0,\sqrt{\varepsilon})}\frac{1}{|y|^{n-3+\rho}} \le C\varepsilon^{1-\frac{\rho}{2}}\|\phi\|_\rho,$$

and

$$\left|\int_{\partial D_\varepsilon\cap O_\varepsilon} Z_i\frac{\partial\phi}{\partial\nu}\right| \le C\|\phi\|_\rho\int_{\partial D_\varepsilon\cap O_\varepsilon}\varepsilon^{\frac{n-2}{2}}\varepsilon^{\frac{3-\rho}{2}} \le C\varepsilon^{1-\frac{\rho}{2}}\|\phi\|_\rho.$$

The second estimate in (6.5) is thus proven.

Proof of (6.6). We directly see that

$$\left|\int_{D_\varepsilon} h_\varepsilon Z_i\right| \le C\left(\int_{I_\varepsilon}\frac{dy}{|y|^{n-2}} + \int_{O_\varepsilon}\frac{1}{(1+|y|^{n+2})}\,dy\right)\|h\|_{**} \le C\|h\|_{**}.$$

Let G_ε denotes the Green's function of D_ε. We have for $x \in D_\varepsilon$

$$\phi_\varepsilon(x) = \underbrace{p\int_{D_\varepsilon} G_\varepsilon(x,y)V^{p-1}\phi_\varepsilon dy}_{:=g_1} - \underbrace{\int_{D_\varepsilon} G_\varepsilon(x,y)h_\varepsilon\,dy}_{:=g_2} - \underbrace{\sum_j c_j\int_{D_\varepsilon} V^{p-1}Z_jG_\varepsilon(x,y)\,dy}_{:=g} \tag{6.9}$$

We claim that

$$|\phi_\varepsilon(x)| \le C\begin{cases}\frac{\|\phi_\varepsilon\|_\rho+\|h_\varepsilon\|_{**}}{|x|^{n-4}} & \text{if } n \ge 5;\\ \frac{\|\phi_\varepsilon\|_\rho+\|h_\varepsilon\|_{**}}{|x|^{\sigma}} & \text{if } n = 4;\\ \|\phi_\varepsilon\|_\rho+\|h_\varepsilon\|_{**} & \text{if } n = 3,\end{cases} \quad \text{for} \quad |x| < 1 \tag{6.10}$$

and

$$|\phi_\varepsilon(x)| \le C\begin{cases}\frac{\|\phi_\varepsilon\|_\rho+\|h_\varepsilon\|_{**}}{1+|x|^2} & \text{if } n \ge 5;\\ \frac{\|\phi_\varepsilon\|_\rho+\|h_\varepsilon\|_{**}}{1+|x|^{2-\sigma}} & \text{if } n = 4;\\ \frac{\|\phi_\varepsilon\|_\rho+\|h_\varepsilon\|_{**}}{1+|x|} & \text{if } n = 3,\end{cases} \quad \text{for} \quad |x| > 1. \tag{6.11}$$

And similarly,

$$|\partial_x \phi_\varepsilon(x)| \le C \begin{cases} \frac{\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**}}{|x|^{n-3}} & \text{if } n \ge 5; \\ \frac{\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**}}{|x|^{\sigma+1}} & \text{if } n = 4; \\ \|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**} \text{ if } & n = 3, \end{cases} \quad \text{for} \quad |x| < 1 \tag{6.12}$$

and

$$|\partial_x \phi_\varepsilon(x)| \le C \begin{cases} \frac{\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**}}{1+|x|^3} & \text{if } n \ge 5; \\ \frac{\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**}}{1+|x|^{3-\sigma}} & \text{if } n = 4; \\ \frac{\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**}}{1+|x|^2} & \text{if } \quad n = 3, \end{cases} \quad \text{for} \quad |x| > 1. \tag{6.13}$$

Assume for the moment the validity of these estimates. Since ρ is arbitrarily small and $\|\phi_\varepsilon\|_\rho = 1$, the estimates above imply that that $\|\phi_\varepsilon\|_{L^\infty(B(0,R_1)\setminus B(0,R_2))} > \gamma$ for certain $R_1 > R_2 > 0$ and $\gamma > 0$ independent of ε. Then local elliptic estimates and the bounds above yield that, up to a subsequence, ϕ_ε converges uniformly over compacts of $\mathbb{R}^n$ to a nontrivial solution $\tilde\phi$ of

$$\Delta\tilde\phi + p|Q|^{p-1}\tilde\phi = 0, \tag{6.14}$$

which besides satisfies

$$|\tilde\phi(x)| \le C|x|^{-\beta}, \quad \beta = 2, \quad \text{if} \quad n \ge 5, \quad \beta = 2 - \sigma, \quad \text{if} \quad n = 4 \tag{6.15}$$

In dimension $n = 3$ this means $|\tilde\phi(x)| \le C|x|^{2-n}$. In higher dimension, a bootstrap argument of $\tilde\phi$ solution of (6.14), using estimate (6.15), gives $|\tilde\phi(x)| \le C|x|^{2-n}$. Thanks to non degenerate result in [19], this implies that $\tilde\phi$ is a linear combination of the functions z_j, defined in (2.15), (2.16), (2.17), (2.18) and (2.19). On the other hand, dominated convergence Theorem gives that the orthogonality conditions $\int_{D_\varepsilon} \phi_\varepsilon V^{p-1} Z_j = 0$ pass to the limit, thus getting

$$\int_{\mathbb{R}^n} |Q|^{p-1} z_j \tilde\phi = 0 \quad \text{for all} \quad j = 0, 1, \ldots, 3n-1.$$

Hence the only possibility is that $\tilde\phi \equiv 0$, which is a contradiction which yields the proof of $\|\phi_\varepsilon\|_\rho \to 0$. Moreover, we observe that

$$\|\phi_\varepsilon\|_* \le C(\|h_\varepsilon\|_{**} + \|\phi_\varepsilon\|_\rho),$$

hence $\|\phi_\varepsilon\|_* \to 0$.

We shall show the validity of (6.10) and (6.11). To get (6.12) and (6.13), one proceeds in a similar way, using the fact the function ϕ_ε is of class C^1 and

$$\begin{aligned} \partial_{x_s}\phi_\varepsilon(x) = p \int_{D_\varepsilon} \partial_{x_s} G_\varepsilon(x,y) V^{p-1}\phi_\varepsilon dy - \int_{D_\varepsilon} \partial_{x_s} G_\varepsilon(x,y) h_\varepsilon \, dy \\ - \sum_j c_j \int_{D_\varepsilon} V^{p-1} Z_j \partial_{x_s} G_\varepsilon(x,y) \, dy, \qquad x \in D_\varepsilon. \end{aligned}$$

Using the definitions of the norm in (5.11), we get that, for $|x| \le 1$,

$$|g_2(x)| \le C\|h_\varepsilon\|_{**} \left(\int_{I_\varepsilon} \frac{1}{|x-y|^{n-2}} \frac{1}{|y|^{n-2}} \, dy + \int_{O_\varepsilon} \frac{1}{|x-y|^{n-2}} \frac{1}{1+|y|^4} \, dy \right)$$

$$\leq C \begin{cases} \frac{\|h_\varepsilon\|_{**}}{|x|^{n-4}} & \text{if } n \geq 5; \\ \frac{\|h_\varepsilon\|_{**}}{|x|^{\sigma}} & \text{if } n = 4; \\ \|h_\varepsilon\|_{**} \text{ if } & n = 3, \end{cases} \tag{6.16}$$

as consequence of (8.9), in Lemma 8.3. Consider now $|x| > 1$. In this region we have

$$|g_2(x)| \leq C \begin{cases} \frac{\|h_\varepsilon\|_{**}}{1+|x|^2} & \text{if } n \geq 5; \\ \frac{\|h_\varepsilon\|_{**}}{1+|x|^{2-\sigma}} & \text{if } n = 4; \\ \frac{\|h_\varepsilon\|_{**}}{1+|x|} & \text{if } \quad n = 3, \end{cases} \tag{6.17}$$

as consequence of (8.10), in Lemma 8.3. Arguing similarly, and using (6.7), we see that, for $|x| < 1$,

$$\left| \sum_j c_j \int_{D_\varepsilon} V^{p-1} Z_j G_\varepsilon(x,y)\, dy \right| \leq C \begin{cases} \frac{\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**}}{|x|^{n-4}} & \text{if } n \geq 4; \\ \|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**} \text{ if } & n = 3, \end{cases} \tag{6.18}$$

and, for $|x| > 1$,

$$\left| \sum_j c_j \int_{D_\varepsilon} V^{p-1} Z_j G_\varepsilon(x,y)\, dy \right| \leq C \begin{cases} \frac{\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**}}{1+|x|^2} & \text{if } n \geq 5; \\ \frac{\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**}}{1+|x|^{2-\sigma}} & \text{if } n = 4; \\ \frac{\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**}}{1+|x|} \text{ if } & n = 3, \end{cases} \tag{6.19}$$

In order to estimate g_1, we consider first $n \geq 5$. For $|x| \leq 1$, we use (8.10) to get

$$|g_1(x)| \leq C\|\phi_\varepsilon\|_\rho \left(\int_{I_\varepsilon} \frac{1}{|x-y|^{n-2}} \frac{dy}{|y|^{n-4+\rho}} + \int_{O_\varepsilon} \frac{1}{|x-y|^{n-2}} \frac{dy}{1+|y|^{6+\rho}} \right) C \frac{\|\phi_\varepsilon\|_\rho}{|x|^{n-4}}, \tag{6.20}$$

and (8.9) to get, for $|x| > 1$

$$|g_1(x)| \leq C \frac{\|\phi_\varepsilon\|_\rho}{1+|x|^2}, \tag{6.21}$$

In a similar way, we have, for $|x| > 1$,

$$|g_1(x)| \leq C \frac{\|\phi_\varepsilon\|_\rho}{|x|^\sigma} \quad \text{if} \quad n = 4, \quad |g_1(x)| \leq C\|\phi_\varepsilon\|_\rho \quad \text{if} \quad n = 3 \tag{6.22}$$

and

$$|g_1(x)| \leq C \frac{\|\phi_\varepsilon\|_\rho}{1+|x|^{2-\sigma}} \quad \text{if} \quad n = 4, \quad |g_1(x)| \leq C \frac{\|\phi_\varepsilon\|_\rho}{1+|x|} \quad \text{if} \quad n = 3. \tag{6.23}$$

Collecting together estimates (6.16)–(6.23), we obtain the validity of (6.10) and (6.11).

Step 2: The existence of solution to (5.10). To do this, let us consider the space

$$H = \left\{ \phi \in H_0^1(D_\varepsilon) \mid \int_{D_\varepsilon} V^{p-1} Z_j \phi = 0, \ \forall\, j = 0, 1, \ldots, 3n-1 \right\}$$

endowed with the usual inner product $[\phi,\psi]=\int_{D_\varepsilon}\nabla\phi\nabla\psi$. Problem (5.10) expressed in weak form is equivalent to that of finding a $\phi\in H$ such that

$$[\phi,\psi]=\int_{D_\varepsilon}\big(pV^{p-1}\phi-h\big)\ \psi \qquad \forall\psi \quad \in H.$$

With the aid of Riesz's representation theorem, this equation gets rewritten in H in the operational form

$$\phi=L_\varepsilon(\phi)+\tilde h \tag{6.24}$$

with certain $\tilde h\in H$ which depends linearly in h and where L_ε is a compact operator in H. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation $\phi=T_\varepsilon(\phi)$ has only the zero solution in H. Assume it has a nontrivial solution $\phi=\phi_\varepsilon$, which with no loss of generality may be taken so that $\|\phi_\varepsilon\|_*=1$. But for what we proved before, necessarily $\|\phi_\varepsilon\|_*\to 0$. This is certainly a contradiction that proves that this equation only has the trivial solution in H. We conclude then that for each h, problem (5.10) admits a unique solution. Standard arguments give then the validity of (5.17).

We next analysis the dependence of the solution ϕ to (5.10) on the parameters $A'=(d,\tau,a,\theta)$. Let us define $A'=(A_1,A_2,\dots,A_{3n})$ the components of the vector A'. Let us differential ϕ with respect to A_s, for some $s=1,\dots,3n$. We set formally $Z=\frac{\partial}{\partial A_s}\phi$. Then Z satisfies the following equation

$$\Delta Z+pV^{p-1}Z=-p\partial_{A_s}(V^{p-1})\phi+\sum_j e_jV^{p-1}Z_j+c_j\partial_{A_s}(V^{p-1}Z_j)\quad \text{in}\quad D_\varepsilon$$

Here $e_j=\partial_{A_s}c_j$. Besides, from differentiating the orthogonality condition $\int_{D_\varepsilon}V^{p-1}Z_j\phi dy=0$, we get

$$\int_{D_\varepsilon}\partial_{A_s}(V^{p-1}Z_j)\phi dy+\int_{D_\varepsilon}V^{p-1}Z_jZdy=0.$$

Let us consider constants b_i such that

$$\int_{D_\varepsilon}V^{p-1}Z_jZ-\sum_i b_i\int_{D_\varepsilon}V^{p-1}Z_jZ_i=0. \tag{6.25}$$

These relations amount to

$$\sum_i b_i\int_{D_\varepsilon}V^{p-1}Z_jZ_i=\int_{D_\varepsilon}V^{p-1}Z_jZ.$$

Since this system is diagonal dominant with uniformly bounded coefficients, we use it is uniquely solvable and that $b_i=O(\|\phi\|_*)$. Let us set $\eta=Z-\sum\limits_i b_iZ_j$, thus $\eta\in H_0^1(D_\varepsilon)$ and $\int_{D_\varepsilon}V^{p-1}Z_j\eta=0$ for all j. On the other hand, we have that

$$\Delta\eta+pV^{p-1}\eta=f+\sum_j e_jV^{p-1}Z_j\quad \text{in } D_\varepsilon, \tag{6.26}$$

where $e_j=\frac{\partial}{\partial A_s}c_j$ and

$$f=\sum_j b_j(-(\Delta+pV^{p-1})Z_j+c_j\partial_{A_s}(V^{p-1}Z_j)-p\partial_{A_s}(V^{p-1}\phi), \tag{6.27}$$

Thus we have that $\eta=T_\varepsilon(f)$. Moreover, we easily see that $\|\phi\partial_{A_s}(V^{p-1})\|_{**}\le C\|\phi\|_*$. On the other hand $|\partial_{A_s}(V^{p-1}Z_i(x))|\le C|x|^{-n-4}$, hence $\|c_i\partial_{A_s}V^{p-1}Z_i\|_{**}\le C\|h\|_{**}$ since we have that $c_i=O(\|h\|_{**})$. We conclude that $\|f\|_{**}\le C\|h\|_{**}$. Reciprocally, if we define $Z=T_\varepsilon(f)+$

$\sum_j e_j V^{p-1} Z_j$, with b_j given by relations (6.25) and f by (6.27), we check that indeed $Z = \partial_{A_s}\phi$. In fact Z depends continuously on the parameters A' and h for the norm $\| \, \|_*$, and $\|Z\|_* \le C\|h\|_{**}$ for parameters in the considered region. In other words, we proved that $(d,\tau,a,\theta) \mapsto T_\varepsilon$ is of class C^1 in $\mathcal{L}(L^\infty_{**}, L^\infty_*)$ and, for instance,

$$(D_{A_s}T_\varepsilon)(h) = T_\varepsilon(f) + \sum b_j Z_j,$$

where f is given by (6.27) and b_j by (6.25). This concludes the proof. □

7. The non-linear Problem: proof of Proposition 5.2

Proof of Proposition 5.2: We write the equation in (5.19) as

$$\Delta\phi + pV^{p-1}\phi = -N(\phi) - E + \sum_j c_j V^{p-1} Z_j \quad \text{in} \quad D_\varepsilon$$

where $N(\phi)$ and E are defined respectively by (5.6) and (5.7). We already showed in (5.13) that $\|E\|_{**} \le C\varepsilon^{\frac{n-2}{2}}$. To estimate $N(\phi)$, it is convenient, and sufficient for our purposes, to assume $\|\phi\|_* < 1$. Note that, if $n \le 6$, then $p \ge 2$ and we can estimate

$$|N(\phi)| \le C|V|^{p-2}|\phi|^2 \quad \text{and hence} \quad |N(\phi)|(x) \le C \begin{cases} \frac{\|\phi\|_*^2}{|x|^2} & \text{if } n \ge 4; \\ \|\phi\|_*^2 & \text{if } n = 3, \end{cases} \quad \text{for} \quad |x| < 1,$$

and

$$|N(\phi)|(x) \le C \begin{cases} \frac{\|\phi\|_*^2}{1+|x|^4} & \text{if } n \ge 5; \\ \varepsilon^{-\frac{\sigma}{2}} \frac{\|\phi\|_*^2}{1+|x|^{4-\sigma}} & \text{if } n = 4; \\ \frac{\|\phi\|_*^2}{1+|x|^4} & \text{if } n = 3, \end{cases} \quad \text{for} \quad |x| > 1,$$

Assume now that $n > 6$. If $|\phi| \ge \frac{1}{2}V$ we have

$$|N(\phi)| \le C|\phi|^p, \quad \text{and thus} \quad |N(\phi)(x)| \le C \begin{cases} \varepsilon^p \frac{\|\phi\|_*}{|x|^{n-2}} & \text{if } |x| < 1; \\ \varepsilon^p \frac{\|\phi\|_*}{1+|x|^4} & \text{if } |x| > 1 \end{cases}$$

Let us consider now the case $|\phi| \le \frac{1}{2}V$. In this case, we have that $|N(\phi)| \le C|V|^{p-1}|\phi|$, for some constant C. Thus, for $|x| < 1$, we get

$$|N(\phi)| \le C\frac{\varepsilon^2}{|y|^4}\frac{\|\phi\|_*}{|y|^{n-4}} \le C\varepsilon\frac{\|\phi\|_*}{|y|^{n-2}}$$

while for $|x| > 1$,

$$|N(\phi)| \le C\varepsilon^2\frac{\|\phi\|_*}{1+|x|^2} \le C\varepsilon\frac{\|\phi\|_*}{1+|x|^4}.$$

Combining these relations we get

$$\|N(\phi)\|_{**} \le \begin{cases} C\|\phi\|_*^2 & \text{if } n \le 6, \, n \ne 4 \\ C\varepsilon^{-\frac{\sigma}{2}}\|\phi\|_*^2 & \text{if } n = 4 \\ C\varepsilon^{-\frac{n-4}{n-2}}\|\phi\|_*^2 & \text{if } n > 6. \end{cases} \tag{7.1}$$

Now, we are in position to prove that problem (5.19) has a unique solution $\phi = \widetilde{\phi} + \widetilde{\psi}$, with

$$\widetilde{\psi} := -T_\varepsilon(E), \tag{7.2}$$

with the required properties. Here T_ε denotes the linear operator defined by Proposition 5.1, namely $\phi = T_\varepsilon(h)$ solves (5.10). We see that problem (5.19) is equivalent to solving a fixed point problem. Indeed $\phi = \tilde{\phi} + \tilde{\psi}$ is a solution of (5.19) if and only if

$$\tilde{\phi} = -T_\varepsilon(N(\tilde{\phi} + \tilde{\psi})) \equiv A_\varepsilon(\tilde{\phi}).$$

We proceed to prove that the operator A_ε defined above is a contraction inside a properly chosen region. Since $\|E\|_* \le C\varepsilon^{\frac{n-2}{2}}$, the result of Proposition 5.2 gives that

$$\|\tilde{\psi}\|_{**} \le C\varepsilon^{\frac{n-2}{2}}$$

and

$$\|N(\tilde{\psi} + \eta)\|_{**} \le \begin{cases} C\|\eta\|_*^2 & \text{if } n \le 6,\ n \ne 4 \\ C\varepsilon^{-\frac{\sigma}{2}}\|\eta\|_*^2 & \text{if } n = 4 \\ C\varepsilon^{-\frac{n-4}{n-2}}\|\eta\|_*^2 & \text{if } n > 6. \end{cases} \tag{7.3}$$

Call

$$F = \{\eta \in H_0^1 \,:\, ||\eta||_* \le R\varepsilon^{\frac{n-2}{2}}\}.$$

From Proposition 5.2 and (7.3) we conclude that, for ε sufficiently small and any $\eta \in F$ we have

$$\|A_\varepsilon(\eta)\|_* \le C\varepsilon^{\frac{n-2}{2}}.$$

If we choose R big enough in the definition of F, we get then that A_ε maps F in itself. Now we will show that the map A_ε is a contraction, for any ε small enough. That will imply that A_ε has a unique fixed point in F and hence problem (5.19) has a unique solution. For any η_1, η_2 in F we have

$$\|A_\varepsilon(\eta_1) - A_\varepsilon(\eta_2)\|_* \le C\|N_\varepsilon(\tilde{\psi} + \eta_1) - N_\varepsilon(\tilde{\psi} + \eta_2)\|_{**},$$

hence we just need to check that N is a contraction in its corresponding norms. By definition of N

$$D_{\bar{\eta}}N_\varepsilon(\bar{\eta}) = p[(V + \bar{\eta})_+^{p-1} - V^{p-1}].$$

Arguing as before, we get $c \in (0, 1)$ such that

$$\|N(\tilde{\psi} + \eta_1) - N(\tilde{\psi} + \eta_2)\|_{**} \le c\|\eta_1 - \eta_2\|_*.$$

This concludes the proof of existence of ϕ solution to (5.19), and the first estimate in (5.20).

The regularity of the map $(d, \tau, a, \theta) \mapsto \phi$ can be proved by standard arguments involving the implicit function, and then we get the estimate (5.21), which can be seen in [19]. $\square$

8. Appendix

Let

$$\mathcal{Z}_0(y) = \frac{n-2}{2}U(y) + \nabla U(y) \cdot y, \quad \mathcal{Z}_\alpha(y) = \frac{\partial}{\partial y_\alpha}U(y),$$

for $\alpha = 1, 2, \cdots, n$. We note that, by symmetry and (8.5) in [19], we have

$$\int_{\mathbb{R}^n} U(y)^{p-1}\mathcal{Z}_0(y)^2 dy = \int_{\mathbb{R}^n} U(y)^{p-1}\mathcal{Z}_\alpha(y)^2 dy = 2^{\frac{n-4}{2}} n(n-2)^2 \frac{\Gamma(\frac{n}{2})^2}{\Gamma(n+2)} := \tilde{c}, \tag{8.1}$$

We have the following results.

Lemma 8.1. *Let the functions z_i be defined in* (2.15)-(2.19)*, and $\mu = \mu_k$ be defined in* (1.10) *and satisfies (1.11). It holds that,*

$$\int_{\mathbb{R}^n} |Q(y)|^{p-1} z_i(y)^2\, dy = (k+1)\tilde{c} + \begin{cases} O(k^{(1-\frac{n}{q})\max\{1,\frac{4}{n-2}\}}), & \text{if} \quad n \geq 4, \\ O(|\log k|^{-1}), & \text{if} \quad n = 3, \end{cases} \tag{8.2}$$

for $i = 0, 1, 2, \cdots, 3n-1$, where $\frac{n}{2} < q < n$ and $\tilde{c}$ is as in (8.1). Moreover, there exists $C > 0$ such that

$$|z_i(y)| \leq C \frac{1}{1+|y|^{n-2}}, \quad \text{for} \quad y \in \mathbb{R}^n. \tag{8.3}$$

Proof. We will give the proof for the case $i = 0$ in (8.2), and the others can be obtained in the same way. Moreover, (8.3) follows directly from the definition of z_i and the results of Proposition 2.1 in [19]. We have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |Q(y)|^{p-1} z_0(y)^2\, dy \\
=& \int_{\mathbb{R}^n} |Q(y)|^{p-1} \Big[\frac{n-2}{2} Q(y) + \nabla Q(y) \cdot y\Big]^2 dy \\
=& \int_{\mathbb{R}^n} |Q(y)|^{p-1} \Big[\Big(\frac{n-2}{2} U(y) + \nabla U(y) \cdot y\Big) \\
&\qquad - \sum_{j=1}^{k} \Big(\frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y\Big) + \pi_0(y)\Big]^2 dy \\
=& \int_{\mathbb{R}^n} |Q(y)|^{p-1} \Big(\frac{n-2}{2} U(y) + \nabla U(y) \cdot y\Big)^2 dy \\
&+ \sum_{j=1}^{k} \int_{\mathbb{R}^n} |Q(y)|^{p-1} \Big(\frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y\Big)^2 dy \\
&+ \int_{\mathbb{R}^n} |Q(y)|^{p-1} |\pi_0(y)|^2 dy \\
&+ 2 \int_{\mathbb{R}^n} |Q(y)|^{p-1} \Big(\frac{n-2}{2} U(y) + \nabla U(y) \cdot y\Big) \\
&\qquad \times \Big[-\sum_{j=1}^{k} \Big(\frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y\Big) + \pi_0(y)\Big] dy \\
&+ 2 \sum_{i \neq j} \int_{\mathbb{R}^n} |Q(y)|^{p-1} \Big(\frac{n-2}{2} U_i(y) + \nabla U_i(y) \cdot y\Big) \Big(\frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y\Big) dy \\
:=& A_1 + A_2 + A_3 + A_4 + A_5.
\end{aligned} \tag{8.4}$$

Next we estimate each term as follows. Let

Estimate of A_1: We have

$$\begin{aligned}
A_1 =& \int_{\mathbb{R}^n} \Big|U(y) - \sum_{j=1}^{k} U_j(y) + \tilde{\phi}(y)\Big|^{p-1} \Big(\frac{n-2}{2} U(y) + \nabla U(y) \cdot y\Big)^2 dy \\
=& \int_{\mathbb{R}^n} \Big[|U(y)|^{p-1} + \sum_{j=1}^{k} |U_j(y)|^{p-1} + |\tilde{\phi}(y)|^{p-1} + |U(y)|^{\gamma} \Big|\sum_{j=1}^{k} U_j(y) + \tilde{\phi}(y)\Big|^{p-1-\gamma}\Big] \mathcal{Z}_0(y)^2 dy
\end{aligned}$$

$$
\begin{aligned}
&=\int_{\mathbb{R}^n}|U(y)|^{p-1}\mathcal{Z}_0(y)^2dy+\sum_{j=1}^k\int_{\mathbb{R}^n}|U_j(y)|^{p-1}\mathcal{Z}_0(y)^2dy\\
&\quad+\int_{\mathbb{R}^n}\Big[|\tilde\phi(y)|^{p-1}+|U(y)|^{\gamma}|\sum_{j=1}^kU_j(y)+\tilde\phi(y)|^{p-1-\gamma}\Big]\mathcal{Z}_0(y)^2dy.
\end{aligned}
$$

Since $\sum_{j=1}^k\int_{\mathbb{R}^n}|U_j(y)|^{p-1}\mathcal{Z}_0(y)^2dy=O(k\mu_k^2)$, and

$$
\Big|\int_{\mathbb{R}^n}\Big[|\tilde\phi(y)|^{p-1}+|U(y)|^{\gamma}|\sum_{j=1}^kU_j(y)+\tilde\phi(y)|^{p-1-\gamma}\Big]\mathcal{Z}_0(y)^2dy\Big|\le C\begin{cases}k^{(1-\frac{n}{q})\frac{4}{n-2}}, & \text{if } n\ge 4,\\ |\log k|^{-4}, & \text{if } n=3.\end{cases}
$$

Thus

$$
A_1=\tilde c+\begin{cases}O(k^{(1-\frac{n}{q})\frac{4}{n-2}}), & \text{if } n\ge 4,\\ O(|\log k|^{-4}), & \text{if } n=3.\end{cases}\tag{8.5}
$$

Here $\tilde c$ is as in (8.1).

Estimate of A_2: We have

$$
\begin{aligned}
A_2&=\int_{\mathbb{R}^n}\Big|U(y)-\sum_{j=1}^kU_j(y)+\tilde\phi(y)\Big|^{p-1}\Big(\frac{n-2}{2}U_j(y)+\nabla U_j(y)\cdot y\Big)^2dy\\
&=\int_{\mathbb{R}^n}\Big[|U(y)|^{p-1}+\sum_{j=1}^k|U_j(y)|^{p-1}+|\tilde\phi(y)|^{p-1}+|U(y)|^{\gamma}|\sum_{j=1}^kU_j(y)+\tilde\phi(y)|^{p-1-\gamma}\Big]\\
&\qquad\times\Big(\frac{n-2}{2}U_j(y)+\nabla U_j(y)\cdot y\Big)^2dy\\
&=\int_{\mathbb{R}^n}|U(y)|^{p-1}\Big(\frac{n-2}{2}U_j(y)+\nabla U_j(y)\cdot y\Big)^2dy\\
&\quad+\sum_{j=1}^k\int_{\mathbb{R}^n}|U_j(y)|^{p-1}\Big(\frac{n-2}{2}U_j(y)+\nabla U_j(y)\cdot y\Big)^2dy\\
&\quad+\int_{\mathbb{R}^n}\Big[|\tilde\phi(y)|^{p-1}+|U(y)|^{\gamma}|\sum_{j=1}^kU_j(y)+\tilde\phi(y)|^{p-1-\gamma}\Big]\Big(\frac{n-2}{2}U_j(y)+\nabla U_j(y)\cdot y\Big)^2dy.
\end{aligned}
$$

Observe that

$$
\int_{\mathbb{R}^n}|U(y)|^{p-1}\Big(\frac{n-2}{2}U_j(y)+\nabla U_j(y)\cdot y\Big)^2dy=O(\mu_k^{n-2}),
$$

$$
\begin{aligned}
&\sum_{j=1}^k\int_{\mathbb{R}^n}|U_j(y)|^{p-1}\Big(\frac{n-2}{2}U_j(y)+\nabla U_j(y)\cdot y\Big)^2dy\\
=&(n-2)^2\alpha_n^{p+1}\sum_{j=1}^k\mu_k^2\int_{\mathbb{R}^n}\frac{\mu_k^2+|y-\xi_j|^2-2\sum_{i=1}^ny_i(y-\xi_j)_i}{(\mu_k^2+|y-\xi_j|^2)^{n+2}}dy\\
=&(n-2)^2\alpha_n^{p+1}\sum_{j=1}^k\int_{\mathbb{R}^n}\frac{1+|z|^2-2\sum_{i=1}^nz_i(z-\frac{\xi_j}{\mu_k})_i}{(1+|z|^2)^{n+2}}dy
\end{aligned}
$$

$$
\begin{aligned}
=&(n-2)^2\alpha_n^{p+1}\sum_{j=1}^k\int_{\{|z|\le\frac{1}{2\mu_k}\}}\frac{1+|z|^2-2\sum_{i=1}^n z_i(z-\frac{\xi_j}{\mu_k})_i}{(1+|z|^2)^{n+2}}dy\\
&+(n-2)^2\alpha_n^{p+1}\sum_{j=1}^k\int_{\{|z|\ge\frac{1}{2\mu_k}\}}\frac{1+|z|^2-2\sum_{i=1}^n z_i(z-\frac{\xi_j}{\mu_k})_i}{(1+|z|^2)^{n+2}}dy\\
=&(n-2)^2\alpha_n^{p+1}\sum_{j=1}^k\int_{\mathbb{R}^n}\frac{1-|z|^2}{(1+|z|^2)^{n+2}}dy+O(\mu_k^{n+2})\\
=&k\tilde{c}+O(\mu_k^{n+2}).
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^n}\Big[|\tilde{\phi}(y)|^{p-1}+|U(y)|^{\gamma}|\sum_{j=1}^k U_j(y)+\tilde{\phi}(y)|^{p-1-\gamma}\Big]\Big(\frac{n-2}{2}U_j(y)+\nabla U_j(y)\cdot y\Big)^2dy\Big|\\
\le& C\begin{cases} k^{(1-\frac{n}{q})\frac{4}{n-2}}, & \text{if } n\ge 4,\\ |\log k|^{-4}, & \text{if } n=3,\end{cases}.
\end{aligned}
$$

Therefore, we obtain

$$
A_2=k\tilde{c}+\begin{cases} O(k^{(1-\frac{n}{q})\frac{4}{n-2}}), & \text{if } n\ge 4,\\ O(|\log k|^{-4}), & \text{if } n=3.\end{cases}\tag{8.6}
$$

Estimate of A_3, A_4, A_5: We have

$$
|A_3|\le C\begin{cases} k^{2(1-\frac{n}{q})}, & \text{if } n\ge 4,\\ |\log k|^{-2}, & \text{if } n=3,\end{cases},\quad |A_4|\le C\begin{cases} k^{1-\frac{n}{q}}, & \text{if } n\ge 4,\\ |\log k|^{-1}, & \text{if } n=3,\end{cases}\quad |A_5|\le Ck\mu_k^2.\tag{8.7}
$$

Thus (8.2) holds for $i=0$ follows from (8.4)-(8.7). □

Lemma 8.2. *Let the functions z_j be defined in* (2.15)-(2.19)*, and $\mu=\mu_k$ be defined in* (1.10) *and satisfies (1.11). It holds that, for $i\neq j$,*

(8.8)

$$
\int_{\mathbb{R}^n}|Q(y)|^{p-1}z_i(y)z_j(y)\,dy=\begin{cases}
(k+1)\tilde{c}+\begin{cases} O(k^{1-\frac{n}{q}}), & \textit{if } n\ge 4,\\ O(|\log k|^{-1}), & \textit{if } n=3,\end{cases} & \textit{for } i=1,\ j=n+2,\\
(k+1)\tilde{c}+\begin{cases} O(k^{1-\frac{n}{q}}), & \textit{if } n\ge 4,\\ (|\log k|^{-1}), & \textit{if } n=3,\end{cases} & \textit{for } i=2,\ j=n+3,\\
O(\mu^{\frac{n-2}{2}}), & \textit{otherwise},
\end{cases}
$$

where $q\in(\frac{n}{2},n)$ and $\tilde{c}$ is a positive constant, which is defined in (8.1).

Proof. We will only consider the case $i=1, j=n+2$, and the case $i=2, j=n+3$ can be proved in a similar way. Moreover, we omit the proof for the other cases, which can be obtained easily by using the definition of z_i and the symmetry. We have

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|Q(y)|^{p-1}z_1(y)z_{n+2}(y)\,dy\\
=&\int_{\mathbb{R}^n}|Q(y)|^{p-1}z_1(y)\Big(-2y_1z_0(y)+|y|^2z_1(y)\Big)dy\\
=&-2\int_{\mathbb{R}^n}|Q(y)|^{p-1}y_1\frac{\partial Q(y)}{\partial y_1}\Big(\frac{n-2}{2}Q(y)+\nabla Q(y)\cdot y\Big)dy+\int_{\mathbb{R}^n}|Q(y)|^{p-1}|y|^2\Big(\frac{\partial Q(y)}{\partial y_1}\Big)^2dy
\end{aligned}
$$

$:=L_1+L_2.$

Direct computations give

$$L_1 =(k+1)\int_{\mathbb{R}^n}(n-2)^2\alpha_n^{p+1}\frac{y_1^2(1-|y|^2)}{(1+|y|^2)^{n+2}}dz+\begin{cases} O(k^{1-\frac{n}{q}}), & \text{if } \ n\geq 4,\\ O(|\log k|^{-1}), & \text{if } \ n=3,\end{cases}$$

and

$$L_2 =(k+1)\int_{\mathbb{R}^n}(n-2)^2\alpha_n^{p+1}\frac{y_1^2|y|^2}{(1+|y|^2)^{n+2}}dz+\begin{cases} O(k^{1-\frac{n}{q}}), & \text{if } \ n\geq 4,\\ O(|\log k|^{-1}), & \text{if } \ n=3.\end{cases}$$

Therefore, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n}|Q(y)|^{p-1}z_1(y)z_{n+2}(y)\,dy\\
=&(k+1)(n-2)^2\alpha_n^{p+1}\int_{\mathbb{R}^n}\frac{y_1^2}{(1+|y|^2)^{n+2}}dz+\begin{cases} O(k^{1-\frac{n}{q}}), & \text{if } \ n\geq 4,\\ O(|\log k|^{-1}), & \text{if } \ n=3,\end{cases}\\
=&(k+1)\int_{\mathbb{R}^n}U(y)^{p-1}\mathcal{Z}_1(y)^2dy+O(k\mu_k^{\frac{n-2}{2}})+\begin{cases} O(k^{1-\frac{n}{q}}), & \text{if } \ n\geq 4,\\ O(|\log k|^{-1}), & \text{if } \ n=3,\end{cases}\\
=&(k+1)\tilde{c}+\begin{cases} O(k^{1-\frac{n}{q}}), & \text{if } \ n\geq 4,\\ O(|\log k|^{-1}), & \text{if } \ n=3,\end{cases}
\end{aligned}$$

where $\mathcal{Z}_1(y)=\frac{\partial U(y)}{\partial y_1}$. ☐

Lemma 8.3. *For any constant $a>0$, there exists $C>0$ such that*

$$\int_{\mathbb{R}^n}\frac{1}{|y-z|^{n-2}}\,\frac{1}{(1+|z|)^{2+a}}\,dz\leq\frac{C}{1+|y|^a}. \tag{8.9}$$

For any $0<b<n$, there exists a constant $C>0$ such that

$$\int_{B(0,1)}\frac{1}{|y-z|^{n-2}}\,\frac{1}{|z|^{n-b}}\,dz\leq\frac{C}{|y|^{n-2-b}}. \tag{8.10}$$

Proof. Proof of (8.9). We just need to give the estimate for $|y|\geq 2$. Let $d=\frac{1}{2}|y|$. For $z\in B_d(0)$, we have $|z|\leq d$ and $|y-z|\geq|y|-|z|\geq d$, then

$$\begin{aligned}
\int_{B_d(0)}\frac{1}{|y-z|^{n-2}}\,\frac{1}{(1+|z|)^{2+a}}\,dz\leq&\frac{C}{d^{n-2}}\int_{B_d(0)}\frac{1}{(1+|z|)^{2+a}}\,dz\\
\leq&\frac{C}{d^{n-2}}d^{n-2-a}\leq\frac{C}{d^a}.
\end{aligned}$$

For $z\in B_d(y)$, we have $|y-z|\leq d$ and $|z|=|y-(y-z)|\geq|y|-|y-z|\geq d$, we then have

$$\int_{B_d(y)}\frac{1}{|y-z|^{n-2}}\,\frac{1}{(1+|z|)^{2+a}}\,dz\leq\frac{C}{d^{2+a}}\int_{B_d(y)}\frac{1}{|y-z|^{n-2}}\,dz\leq\frac{C}{d^a}.$$

For $z\in\mathbb{R}^n\backslash(B_d(0)\cup B_d(y))$, we have $|z-y\geq\frac{1}{2}|y|$ and $|z|\geq\frac{1}{2}|y|$. If $|z|\geq 2|y|$, we have $|z-y|\geq|z|-|y|\geq\frac{1}{2}|z|$. Thus

$$\frac{1}{|y-z|^{n-2}}\,\frac{1}{(1+|z|)^{2+a}}\leq\frac{C}{|z|^{n-2}(1+|z|)^{2+a}}. \tag{8.11}$$

If $|z| \le 2|y|$, then

$$\frac{1}{|y-z|^{n-2}}\,\frac{1}{(1+|z|)^{2+a}} \le \frac{C}{|y|^{n-2}(1+|z|)^{2+a}} \le \frac{C'}{|z|^{n-2}(1+|z|)^{2+a}}. \tag{8.12}$$

From (8.11) and (8.12), we get that for $z \in \mathbb{R}^n \backslash (B_d(0) \cup B_d(y))$,

$$\frac{1}{|y-z|^{n-2}}\,\frac{1}{(1+|z|)^{2+a}} \le \frac{C}{|z|^{n-2}(1+|z|)^{2+a}}.$$

Then

$$\int_{\mathbb{R}^n \backslash (B_d(0)\cup B_d(y))} \frac{1}{|y-z|^{n-2}}\,\frac{1}{(1+|z|)^{2+a}}\,dz \le \int_{\mathbb{R}^n \backslash (B_d(0)\cup B_d(y))} \frac{C}{|z|^{n-2}(1+|z|)^{2+a}}\,dz \le \frac{C}{d^a}.$$

Proof of (8.10). Assume $y = r\hat{y}$, with $r = |y|$. Direct computation gives

$$\int_{B(0,1)} \frac{1}{|y-z|^{n-2}}\,\frac{1}{|z|^{n-b}}\,dz = \frac{1}{r^{n-2}} \int_{B(0,1)} \frac{1}{|\hat{y}-\frac{z}{r}|^{n-2}}\,\frac{1}{|z|^{n-b}}\,dz$$

$$(w = \frac{z}{r})$$

$$= \frac{1}{r^{n-2-b}} \int_{B(0,\frac{1}{r})} \frac{1}{|\hat{y}-w|^{n-2}}\,\frac{1}{|w|^{n-b}}\,dz \le \frac{C}{|y|^{n-2-b}},$$

where $C = \sup_{e\in S^n} \int_{\mathbb{R}^n} \frac{1}{|e-w|^{n-2}}\,\frac{1}{|w|^{n-b}}\,dz$. □

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